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ABSTRACT. We generalize biharmonic maps between Riemannian manifolds into the case of the domain being V-manifolds. We obtain the first and second variations of biharmonic maps on V-manifolds. Since a biharmonic map from a compact V-manifold into a Riemannian manifold of nonpositive curvature is harmonic, we construct a biharmonic non-harmonic map into a sphere. We also show that under certain condition the biharmonic property of f implies the harmonic property of f. We finally discuss the composition of biharmonic maps on V-manifolds.

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1. Introduction. Following Eells, Sampson, and Lemaire's tentative ideas [7, 8, 9], Jiang first discussed biharmonic (or 2-harmonic) maps between Riemannian manifolds in his two articles [10, 11] in China in 1986, which gives the conditions for biharmonic maps. A biharmonic map $f : M \to N$ between Riemannian manifolds is the smooth critical point of the bi-energy functional

$$E_2(f) = \int_M \left\|(d+d^*)f\right\|^2 * 1 = \int_M \left\|\tau(f)\right\|^2 * 1, \tag{1.1}$$

where $*1$ is the volume form on M, the tension field $\tau(f) = (\hat{D}df)(e_i,e_i) (= (\hat{D}_{e_i}df)(e_i))$, $\{e_i\}$ is the local frame of a point p in M. Biharmonic maps are the extensions of harmonic maps, and their study provides a source in partial differential equations, differential geometry, and analysis. After Jiang, Chiang, and Sun have studied biharmonic maps in two papers [6, 14]. Chiang also studied harmonic maps and biharmonic maps of two different kinds of singular spaces: V-manifolds [3, 4] and spaces with conical singularities (with Andrea Ratto [5]).

In this paper, we generalize the notion of a biharmonic map to the case that the domain of f is a V-manifold due to Satake in [1, 12, 13]. A (C^∞) V-manifold $(M, \mathcal{F})$ consists of a Hausdorff space M with an atlas $\mathcal{F}$ of V-charts satisfying the following conditions:

(i) If $\{\tilde{U}, G, \pi\}$ and $\{\tilde{U}\ , G\ , \pi\ \}$ are two V-charts in $\mathcal{F}$ over U, U , respectively, in M with $U \subset U$, then there exists an injection $\lambda : \{U, G, \pi\} \to \{U, G\ , \pi\ \}$.

(ii) The supports of V-charts in $\mathcal{F}$ form a basis for open sets in M.

Take a chart $\{\tilde{U}, G, \pi\} \in \mathcal{F}$ such that $p \in \pi(\tilde{U})$ and choose $\tilde{p} \in \tilde{U}$ such that $\sigma\tilde{p} = \tilde{p}$. The isotropic subgroup $G_{\tilde{p}}$ of G at $\tilde{p}$ is the set of all $\sigma \in G$ such that $\sigma\tilde{p} = \tilde{p}$. So $G_{\tilde{p}}$ is called the *isotropic group* of p. The singular set $\mathbb{S}$ of M consists of all singular points of M, that is, the points of M with nontrivial isotropy groups. (For example, S^2/Z_3 is a compact V-manifold with two singular points.) The main difficulties of this paper arise from the complicated behavior of the singular locus of V-manifolds, and therefore a different method than the usual one is required. In fact, this article is the extension of Chiang's previous two papers [3, 4].

We derive the first variations of biharmonic maps in Theorem 2.2, and give the definition for biharmonic maps on V-manifolds. We show that a biharmonic map from a compact V-manifold into a Riemannian manifold of nonpositive curvature is a harmonic map in Theorem 2.4. Then we construct a biharmonic non-harmonic map from a V-manifold into a sphere in Section 2. We obtain the second variations of biharmonic maps in Theorem 3.1. If $d^2/dt^2 E_2(f_t)|_{t=0} \geq 0$, then f is a stable biharmonic map. In Theorem 3.3, we show that if a stable biharmonic map from a compact V-manifold M into a Riemannian manifold N of positive curvature satisfies the conservation law, then f must be a harmonic map. In Theorem 3.4, we prove the composition of biharmonic maps on V-manifolds which generalizes Sun's result in [14].

2. Biharmonic maps on V-manifolds. Let $(M,\mathcal{F})$ be a $(\mathbb{C}^\infty)$ V-manifold, and U be an open subset of M. By a V-chart on M over U we mean a system $\{\tilde{U},G,\pi\}$ consisting of (1) a connected open subset $\tilde{U}$ of $\mathbb{R}^m$, (2) a finite group G of diffeomorphisms of $\tilde{U}$, with the set of fixed points of codimension ≥ 2, and (3) a continuous map of $\pi : \tilde{U} \to U$ such that $\pi \circ \sigma = \pi$ for $\sigma \in G$ and such that π induces a homeomorphism of $\tilde{U}/G$ onto U. The set U is called the *support* of V-chart, and π is called the *projection* onto U.

Let $(M,\mathcal{F})$ be a V-manifold and $p \in M$. Take a chart $\{\tilde{U},G,\pi\} \in \mathcal{F}$ such that $p \in \pi(\tilde{U})$ and choose $\tilde{p} \in \tilde{U}$ such that $\pi(\tilde{p}) = p$. The isotropic subgroup $G_{\tilde{p}}$ of G at $\tilde{p}$ is the set of all $\sigma \in G$ such that $\sigma\tilde{p} = \tilde{p}$, and is uniquely determined by p. Therefore, $G_{\tilde{p}}$ is called the *isotropic group* of p. The singular set $\mathbb{S}$ of M consists of all singular points of M, that is, the points of M with nontrivial isotropic groups. Let $(\tilde{x}^1,\dots,\tilde{x}^m)$ be a coordinate system around $\tilde{p}$ and consider the system $\tilde{y}^i = 1/|G_{\tilde{p}}| \sum l_{ij}(\sigma^{-1})\tilde{x}^j \cdot \sigma$ with

$$l_{ij}(\sigma) = \left[\frac{\partial \tilde{x}^i \circ \sigma}{\partial \tilde{x}^j}\right]_{\tilde{p}}, \qquad |G_{\tilde{p}}| = \text{order of } G_{\tilde{p}}. \tag{2.1}$$

Then the $\{\tilde{y}^i\}$ are a new coordinate system around $\tilde{p}$ and $G_{\tilde{p}}$ operates linearly in the $\tilde{y}$-system. After this suitable C^∞ change of coordinates around $\tilde{p}$, $G_{\tilde{p}}$ becomes a finite group of linear transformations. The fixed point set of any $\sigma \in G_{\tilde{p}}$ is the defined linear equations in the $\tilde{y}$, and consequently the fixed point set of $\sigma \in G_{\tilde{p}}$ in $\tilde{U}$ is the intersection of $\tilde{U}$ with a linear space. Therefore, $\pi^{-1}\mathbb{S}$ is locally expressed by a finite union of linear spaces intersected with $\tilde{U}$. Hence $\mathbb{S}$ is a V-submanifold of codimension ≥ 2 of M. Clearly, $M - \mathbb{S}$ is an ordinary manifold.

We fix a V-manifold M with defining atlas $\mathcal{F}$. A *smooth function* $f : (M,\mathcal{F}) \to N$ from M into an ordinary manifold N is defined as follows: for any $\{\tilde{U},G,\pi\} \in \mathcal{F}$ there corresponds an ordinary G-invariant smooth map $f^G_{\tilde{U}} = 1/|G| \sum_{\sigma\in G} f_{\tilde{U}} \circ \sigma : \tilde{U} \to N$ such that $f^G_{\tilde{U}} = f \circ \pi$ and $f^G_{\tilde{U}} = f^G_{\tilde{U}} \circ \lambda$ for any injection $\lambda : \{\tilde{U},G,\pi\} \to \{\tilde{U}\ ,G\ ,\pi\ \}$ where $f_{\tilde{U}} : \tilde{U} \to N$ is an ordinary smooth map.

Put a Riemannian metric $g_{\tilde{U}} = g_{ij}\, d\tilde{x}^i\, d\tilde{x}^j$ on $\tilde{U}$. By taking the G-average if necessary, we can assume that $g_{\tilde{U}}$ is G-invariant. Thus the transformations $\sigma \in G$ are isometries for $g_{\tilde{U}}$. By using the standard partition of unity construction, we can patch all such local invariant metrics together into a global metric tensor field of type $(0,2)$ on the V-manifold M, which we call a Riemannian metric on M.

Let M^m be a compact V-manifold of dimension m with $\mathbb{C}^\infty$ Riemannian metric g, and N^n a ($\mathbb{C}^\infty$) Riemannian manifold of dimension n. By Satake [12, 13], M admits a finite triangulation $T = \cup s_\alpha$ such that each s_α is contained in the support U_α of a V-chart $\{\tilde U_\alpha, G_\alpha, \pi_\alpha\} \in \mathcal{F}$ on M and is the homeomorphic projection of a regular simplex $\tilde s_\alpha$ in $\tilde U_\alpha$. For a smooth map $f : M \to N$, the bi-energy functional of f is defined by

$$E_2(f) = \int_M |\tau(f)|^2 * 1 = \int_{s_\alpha} |\tau(f)|^2 dx_\alpha = \frac{1}{|G_\alpha|}\int_{\tilde s_\alpha} |\tau(\tilde f)|^2 d\tilde x_\alpha, \tag{2.2}$$

where $d\tilde x_\alpha$ denotes the volume form with respect to the G_α-invariant metric g_{ij} in $\tilde U_\alpha$, $\tilde f_\alpha : \tilde U_\alpha \to N$ is the G_α -invariant lift of f. The Green's divergence theorem on a compact V-manifold proved in [3] plays an important role in the proofs of both Theorems 2.2 and 3.1.

In order to compute the Euler-Lagrange equation, we consider a one-parameter family of maps $\{f_t\} \in \mathbb{C}^\infty(M,N)$, $t \in I = (-\epsilon,\epsilon)$, $\epsilon > 0$ such that in the V-chart $\{\tilde U, G, \pi\} \in \mathcal{F}$ over the support U on M, the G-invariant lift $\tilde f_t$ is the endpoint of the segment starting at G-invariant lift $\widetilde{f(x)}$ determined in length and direction by the vector field $\dot{\tilde f}$ along $\tilde f$, and such that $\partial \tilde f_t/\partial t = 0$ and $\bar D_{\tilde e_i}\partial \tilde f_t/\partial t = 0$ outside a compact subset of the interior of $\tilde U$. Choose $\{e_i\}$ being the local frame of a point p in U on M, and $\{\tilde e_i\}$ being the local frame of the lifting point $\tilde p$ in $\tilde U$. Let D, D', $\bar D$, $\hat D$ be the Riemannian connections along TM, TN, $f^{-1}TN$, $T^*M \otimes f^{-1}TN$, and $\tilde D$, $\tilde{\hat D}$ are the Riemannian connections along $T\tilde U$, $T^*\tilde U \otimes f^{-1}TN$ in each $\{\tilde U, G, \pi\} \in \mathcal{F}$ over the support U on M. Also, let $\triangle = \bar D_{\tilde e_k}\bar D_{\tilde e_k} - \bar D_{D_{\tilde e_k}\tilde e_k}$ be the Laplace operator along the cross section of $f^{-1}TN$ in each $\tilde U$, and $V = \partial \tilde f_t/\partial t$. We can compute (2.2) directly, and obtain the following result.

Lemma 2.1.

$$\begin{aligned}\frac{d}{dt}E_2(f_t) = {}& 2\Sigma\frac{1}{|G_\alpha|}\int_{\tilde s_\alpha} \Big\langle \tilde{\hat D}_{\tilde e_i}\tilde{\hat D}_{\tilde e_i} d\tilde f_t\Big(\frac{\partial}{\partial t}\Big) - \tilde{\hat D}_{\tilde D_{\tilde e_i}\tilde e_i} d\tilde f_t\Big(\frac{\partial}{\partial t}\Big), (\tilde{\hat D}_{\tilde e_j} d\tilde f_t)(\tilde e_j)\Big\rangle d\tilde x_\alpha \\ & + 2\Sigma\frac{1}{|G_\alpha|}\int_{\tilde s_\alpha} \Big\langle R^N\Big(d\tilde f_t(\tilde e_i), d\tilde f_t\Big(\frac{\partial}{\partial t}\Big)\Big) d\tilde f_t(\tilde e_i), (\tilde{\hat D}_{\tilde e_j} d\tilde f_t)(\tilde e_j)\Big\rangle d\tilde x_\alpha.\end{aligned} \tag{2.3}$$

Theorem 2.2. *Let $f : (M,\mathcal{F}) \to N$ be a smooth map from a compact V-manifold $(M,\mathcal{F})$ into a Riemannian manifold N. Set $V = \partial \tilde f_t/\partial t$ then*

$$\frac{d}{dt}\Big|_{t=0} E_2(f_t) = 2\Sigma\frac{1}{|G_\alpha|}\int_{\tilde s_\alpha} \Big\langle V, \triangle\tau(\tilde f) + R^N(d\tilde f(\tilde e_i), \tau(\tilde f))\, d\tilde f(\tilde e_i)\Big\rangle d\tilde x_\alpha. \tag{2.4}$$

Proof. For every $t \in I$, let

$$\tilde X = \Big\langle \tilde{\hat D}_{\tilde e_i} d\tilde f_t\Big(\frac{\partial}{\partial t}\Big), \tilde{\hat D}_{\tilde e_j} d\tilde f_t(\tilde e_j)\Big\rangle \tilde e_i, \qquad \tilde Y = \Big\langle d\tilde f_t\Big(\frac{\partial}{\partial t}\Big), \bar D_{\tilde e_i}(\tilde{\hat D}_{\tilde e_j} d\tilde f_t)(\tilde e_j)\Big\rangle (\tilde e_i), \tag{2.5}$$

in each $\{\tilde U, \pi, G\} \in \mathcal{F}$ over the support U on M. By computing the divergence of $\tilde X$ and $\tilde Y$ in each $\tilde U$, and applying Green's divergence theorem to the vector field $\tilde X - \tilde Y$

in each $\tilde{}$ on the compact manifold M in [3], we have

$$\begin{aligned}&\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\left(\tilde{\bar{D}}_{\tilde{e}_i}\tilde{\bar{D}}_{\tilde{e}_i}d\tilde{f}_t\right)\frac{\partial}{\partial t}-\left(\tilde{\bar{D}}_{\tilde{D}_{\tilde{e}_i}\tilde{e}_i}d\tilde{f}_t\right)\frac{\partial}{\partial t},\left(\tilde{\bar{D}}_{\tilde{e}_j}d\tilde{f}_t\right)(\tilde{e}_j)\;d\tilde{x}_\alpha\\&=\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}d\tilde{f}_t\frac{\partial}{\partial t},\bar{D}_{\tilde{e}_k}\bar{D}_{\tilde{e}_k}\left(\tilde{\bar{D}}_{\tilde{e}_j}d\tilde{f}_t\right)(\tilde{e}_j)-\bar{D}_{\tilde{D}_{\tilde{e}_k}\tilde{e}_k}\left(\left(\tilde{\bar{D}}_{\tilde{e}_j}d\tilde{f}_t\right)(\tilde{e}_j)\right)\;d\tilde{x}_\alpha.\end{aligned}\tag{2.6}$$

By the assumption, $\partial\tilde{f}_t/\partial t=0$ and $\bar{D}_{\tilde{e}_i}\partial\tilde{f}_t/\partial t=0$ outside of the compact subset of the interior of each $\tilde{U}$, and substituting (2.6) into (2.3), we get

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}E_2(f_t)=2&\quad\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}d\tilde{f}_t\frac{\partial}{\partial t},\bar{D}_{\tilde{e}_k}\bar{D}_{\tilde{e}_k}\left(\tilde{\bar{D}}_{\tilde{e}_j}d\tilde{f}_t\right)(\tilde{e}_j)\\&\qquad-\bar{D}_{\tilde{D}_{\tilde{e}_k}\tilde{e}_k}\left(\left(\tilde{\bar{D}}_{\tilde{e}_j}d\tilde{f}_t\right)(\tilde{e}_j)\right)\;d\tilde{x}_\alpha\\&+2\quad\frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}R^N\;d\tilde{f}_t(\tilde{e}_i),d\tilde{f}_t\frac{\partial}{\partial t}\;\;d\tilde{f}_t(\tilde{e}_i),\left(\tilde{\bar{D}}_{\tilde{e}_j}d\tilde{f}_t\right)(\tilde{e}_j)\;d\tilde{x}_\alpha.\end{aligned}\tag{2.7}$$

Let $t=0$, and by the symmetry of the Riemannian curvature tensor, we derive (2.4). □

Definition 2.3. A smooth map $f:(M,\mathcal{F})\to N$ from a compact V-manifold M into a Riemannian manifold N is biharmonic if and only if

$$\tau_2(\tilde{f})=\triangle\tau(\tilde{f})+R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))d\tilde{f}(\tilde{e}_i)=0\tag{2.8}$$

in each $\{\tilde{U},G,\pi\}\in\mathcal{F}$ over the support U on M.

A harmonic map $f:M\to N$ on a V-manifold M is obviously a biharmonic map, but a harmonic map is not necessarily a biharmonic map. However, we obtain the following theorem.

Theorem 2.4. *Suppose that M is a compact V-manifold, and N is a Riemannian manifold of nonpositive curvature. If $f:M\to N$ is a biharmonic map, then f is a harmonic map.*

Proof. In each V-chart $\{\tilde{U},G,\pi\}\in\mathcal{F}$ over the support U on M it is calculated by

$$\begin{aligned}e_2(\tilde{f})&=\frac{1}{2}\;\|\tau(\tilde{f})\|^2=\left\langle\bar{D}_{\tilde{e}_k}\tau(\tilde{f}),\bar{D}_{\tilde{e}_k}\tau(\tilde{f})\right\rangle+\left\langle\bar{D}^*\bar{D}\tau(\tilde{f}),\tau(\tilde{f})\right\rangle\\&=\left\langle\bar{D}_{\tilde{e}_k}\tau(\tilde{f}),\bar{D}_{e_k}\tau(\tilde{f})\right\rangle-\left\langle R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))d\tilde{f}(\tilde{e}_i),\tau(\tilde{f})\right\rangle\geq0,\end{aligned}\tag{2.9}$$

because $\tau_2(\tilde{f})=0$ in each $\tilde{U}$ and the Riemannian curvature of N is nonpositive. By Bochner's technique and the assumption $\partial\tilde{f}_t/\partial t=0$ and $\bar{D}_{\tilde{e}_i}\partial\tilde{f}_t/\partial t=0$ outside a compact subset of $\mathrm{int}(\tilde{U})$, we know $\;\tau(\tilde{f})\;^2=\text{const}$, and then substituting into (2.9) we have $\bar{D}_{\tilde{e}_k}(\tau\tilde{f})=0$, for all $k=1,2,\dots,m$ by [7] which implies $\tau(\tilde{f})=0$ in each $\tilde{U}$, that is, f is harmonic on M. □

Since harmonic maps are automatically biharmonic maps when the Riemannian curvature of N is nonpositive, we will find a non-trivial biharmonic map into a sphere. By the concepts of V-manifolds and the similar techniques as [11], we have the following theorem.

Theorem 2.5. *Let $f : (M,\mathcal{F}) \to S^{m+1}$ be nonzero parallel mean curvature isometric embedding, then f is biharmonic if and only if the second fundamental form $B(\tilde{f})$ of $\tilde{f}$ with $B(\tilde{f})\ ^{2} = m = \dim(\tilde{U})$ in each $\tilde{U}$ over the support U on M.*

Example 2.6. In S^{m+1}, the compact hypersurface of its Gauss map being isometric embedding is the Clifford surface (see [15]):

$$M_k^m(1) = S^k\left(\sqrt{\frac{1}{2}}\right) \times S^{m-k}\left(\sqrt{\frac{1}{2}}\right), \quad 0 \le k \le m. \tag{2.10}$$

Let $f : M_k^m(1) \to S^{m+1}$ be the standard embedding. Set

$$M_k^m(1) \ = \frac{S^k(\sqrt{1/2})}{Z_p} \times \frac{S^{m-k}(\sqrt{1/2})}{Z_p}, \tag{2.11}$$

where p, p are prime numbers (p and p could be the same or different). Since both the first and the second terms are compact V-manifolds, the product is also a compact V-manifold. Let $f\ : M_k^m(1)\ \to S^{m+1}$ be a map such that $k \neq m/2$, pick $\tilde{U} = \{(x^0, x^1, \ldots, x^k) \in S^k\sqrt{1/2} : x^i > 0,\ i \textit{ is any of } 0,1,\ldots,k\} \times \{(x^{k+1},\ldots,x^{m+1}) \in S^{m-k}\sqrt{1/2} : x^j > 0,\ j \textit{ is any of } k+1,\ldots,m+1\}$ (if x^i and x^j vary, $\tilde{U}$ is different), and let $\tilde{f}\ : \tilde{U} \to S^{m+1}$ (as part of the standard map $f : S^k\sqrt{1/2} \times S^{m-k}\sqrt{1/2} \to S^{m+1}$) in each $\{\tilde{U}, G, \pi\} \in \mathcal{F}$. So $\tilde{f}$ has parallel second fundamental form, and has parallel mean curvature and $B(\tilde{f}\ ) = k+m-k = m$, $\tau(\tilde{f}\ )\| = |k-(m-k)| = 2k-m \neq 0$. That is, $\tilde{f}$ is biharmonic in $\tilde{U}$ for each $\{\tilde{U}, G, \pi\} \in \mathcal{F}$. Then by Theorem 2.5 f is a nontrivial biharmonic map on $(M,\mathcal{F})$.

3. The stability and composition of biharmonic maps on V-manifolds.

Let M be a compact V-manifold, and N a Riemannian manifold. We continue to use the notations as in the previous sections. By applying the Green's divergence theorem on the compact V-manifold M [3], the concepts of V-manifolds, and the similar techniques in [11], we can have the second variations of biharmonic maps as follows.

Theorem 3.1. *If $f : (M\mathcal{F}) \to N$ is a biharmonic map, then*

$$\begin{aligned}
&\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0} \\
&\quad = \frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha} \|\ \ V + R^N(d\tilde{f}(\tilde{e}_i), V)d\tilde{f}(\tilde{e}_i)\|^2 \, d\tilde{x}_\alpha \\
&\qquad + \frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha} \Big\langle V, (D_{\ d\tilde{f}(\tilde{e}_k)}R^N)(d\tilde{f}(\tilde{e}_k), \tau(\tilde{f}))V \\
&\qquad\qquad + (D_{\ \tau(\tilde{f})}R^N)(d\tilde{f}(\tilde{e}_i), V)\, d\tilde{f}(\tilde{e}_i) + R^N(\tau(\tilde{f}), V)\tau(\tilde{f}) \\
&\qquad\qquad + 2R^N(d\tilde{f}(\tilde{e}_k), V)\bar{D}_{\tilde{e}_k}\tau(\tilde{f}) + 2R^N(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f}))\bar{D}_{\tilde{e}_i}V\Big\rangle d\tilde{x}_\alpha.
\end{aligned} \tag{3.1}$$

Definition 3.2. Let $f : (M,\mathcal{F}) \to N$ be a biharmonic map from a compact V-manifold M into a Riemannian manifold N. If $d^2/dt^2 E_2(f_t)|_{t=0} \ge 0$, then f is a *stable* biharmonic map.

If we look at a harmonic map as a biharmonic map, then it must be stable by the definition of bi-energy since

$$\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0} = \frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha} \| \quad V + R^N(d\tilde{f})((\tilde{e}_i),V)\,d\tilde{f}(\tilde{e}_i)\|^2\,d\tilde{x}_\alpha \geq 0. \tag{3.2}$$

Theorem 3.3. *Let $f:(M,\mathcal{F}) \to N$ be a stable biharmonic map from a compact V-manifold M into a Riemannian manifold N of constant sectional curvature $K > 0$ and f satisfies the conservation law, then f must be a harmonic map.*

Proof. Because N has the constant sectional curvature, the term of $D\ R^N$ of the second variation formula disappears and

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0} = & \frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha} \| \quad V + R^N(df(e_i),V)df(e_i)\|^2\,d\tilde{x}_\alpha \\ &+ \frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\Big\langle V, R^N(\tau(\tilde{f}),V)\tau(\tilde{f}) + 2R^N(d\tilde{f}(\tilde{e}_k),V)\bar{D}_{\tilde{e}_k}\tau(\tilde{f}) \\ &\qquad + 2R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))\bar{D}_{\tilde{e}_i}V\Big\rangle\,d\tilde{x}_\alpha.\end{aligned} \tag{3.3}$$

Take $V = \tau(\tilde{f})$, and notice that f is biharmonic and N has the constant sectional curvature, then by (3.3) we have

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0} = & \frac{4}{|G_\alpha|}\int_{\tilde{s}_\alpha}\Big\langle R^N(d\tilde{f}(\tilde{e}_i),\tau(\tilde{f}))\bar{D}_{\tilde{e}_k}\tau(\tilde{f}),\tau(\tilde{f})\Big\rangle\,d\tilde{x}_\alpha \\ &= 4K \quad \frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\Big[\Big\langle d\tilde{f}(\tilde{e}_k),\tilde{\bar{D}}_{\tilde{e}_k}\tau(\tilde{f})\Big\rangle\|\tau(\tilde{f})\|^2 \\ &\qquad - \Big\langle d\tilde{f}(\tilde{e}_k),\tau(\tilde{f})\Big\rangle\Big\langle\tau(\tilde{f}),\bar{D}_{\tilde{e}_k}\tau(\tilde{f})\Big\rangle\Big]\,d\tilde{x}_\alpha.\end{aligned} \tag{3.4}$$

In each $\tilde{U}_\alpha$, $\tilde{f}$ satisfies the conservation law [2], so

$$\begin{aligned}\Big\langle d\tilde{f}(\tilde{e}_k),\tau(\tilde{f})\Big\rangle &= 0, \\ \Big\langle d\tilde{f}(\tilde{e}_k),\bar{D}_{\tilde{e}_k}\tau(\tilde{f})\Big\rangle = -\Big\langle\bar{D}_{\tilde{e}_k}d\tilde{f}(\tilde{e}_k),\tau(\tilde{f})\Big\rangle &= -\|\tau(\tilde{f})\|^2\end{aligned} \tag{3.5}$$

in each $\tilde{U}$. Substitute (3.5) into (3.4), and f is stable, we have

$$\frac{1}{2}\frac{d^2}{dt^2}E_2(f_t)\Big|_{t=0} = -4K \quad \frac{1}{|G_\alpha|}\int_{\tilde{s}_\alpha}\|\tau(\tilde{f})\|^4\,d\tilde{x}_\alpha \geq 0. \tag{3.6}$$

Therefore, $\tau(\tilde{f}) = 0$ in each $\tilde{s}_\alpha$ of $\tilde{U}_\alpha$, that is, f is harmonic on $(M,\mathcal{F})$.

Let $f:(M,\mathcal{F}) \to M$ be a smooth map from a compact V-manifold $(M,\mathcal{F})$ into a Riemannian manifold and M , and $f_1 : M \to M$ a smooth map from M into another Riemannian manifold M . Then the composition $f_1 \circ f : M \to M$ is a smooth map. Let $D, D\ , \bar{D}, \bar{D}\ \hat{D}, \hat{D}\ , \hat{D}$ be the Riemannian connections on TM, TM , $f^{-1}TM$, $f_1^{-1}TM$, $(f_1\circ f)^{-1}TM$, $T^*M\otimes f^{-1}TM$, $T^*M \otimes f_1^{-1}TM$, $T^*M\otimes(f_1\circ f)^{-1}TM$, respectively, and let $R^M\ (,), R^{f_1^{-1}TM}$ be the Riemannian curvatures on TM , $f^{-1}TM$, respectively. For all $X,Y \in \Gamma(TM)$, we have

$$\bar{D}_X\,d(f_1\circ f)Y = \hat{D}_{df(X)}df_1(Y) + df_1\circ\bar{D}_X df(Y). \tag{3.7}$$

□

Theorem 3.4. *Let $(M,\mathcal{F})$ be a compact V-manifold, and M , M Riemannian manifolds. If $f : M \to M$ is a biharmonic map and $f_1 : M \to M$ is totally geodesic, then the composition $f_1 \circ f : M \to M$ is a biharmonic map.*

Proof. Since f_1 is totally geodesic, that is, $\hat{D}\, df_1 = 0$, so in each $\tilde{U}$ we have $\tau(f_1 \circ \tilde{f}) = df_1 \circ \tau(\tilde{f})$ and

$$
\begin{aligned}
\bar{D} \;{}^{*}\bar{D}\tau(f_1 \circ \tilde{f}) &= \bar{D} \;{}^{*}\bar{D} \;(df_1 \circ \tau(\tilde{f})) \\
&= \bar{D}_{\tilde{e}_k}\bar{D}_{\tilde{e}_k}(df_1 \circ \tau(\tilde{f})) - \bar{D}_{D_{\tilde{e}_k}\tilde{e}_k}(df_1 \circ \tau(\tilde{f})).
\end{aligned} \tag{3.8}
$$

By (3.7) and notice that f_1 is totally geodesic, then

$$
\begin{aligned}
\bar{D}_{\tilde{e}_k}(df_1 \circ \tau(\tilde{f})) &= \bar{D}_{\tilde{e}_k}(df_1 \circ \hat{D}_{\tilde{e}_j} d\tilde{f}(\tilde{e}_j)) \\
&= \left(\hat{D}_{\hat{D}_{\tilde{e}_j} d\tilde{f}(\tilde{e}_k)} df_1\right)(\hat{D}_{\tilde{e}_j} d\tilde{f}(\tilde{e}_j)) + df_1 \circ \bar{D}_{\tilde{e}_k}(\hat{D}_{\tilde{e}_j} d\tilde{f}(\tilde{e}_j)) \\
&= df_1 \circ \bar{D}_{\tilde{e}_k}\tau(\tilde{f}).
\end{aligned} \tag{3.9}
$$

So

$$
\begin{aligned}
&\bar{D}_{\tilde{e}_k}\bar{D}_{\tilde{e}_k}(df_1 \circ \tau(\tilde{f})) = \bar{D}_{\tilde{e}_k}(df_1 \circ \bar{D}_{\tilde{e}_k}\tau(\tilde{f})) = df_1 \circ \bar{D}_{\tilde{e}_k}\bar{D}_{\tilde{e}_k}\tau(\tilde{f}), \\
&\bar{D}_{D_{\tilde{e}_k}\tilde{e}_k}(df_1 \circ \tau(\tilde{f})) = df_1 \circ \bar{D}_{D_{\tilde{e}_k}\tilde{e}_k}\tau(\tilde{f}).
\end{aligned} \tag{3.10}
$$

Substituting (3.10) into (3.8), we get

$$
\bar{D} \;{}^{*}\tau(f_1 \circ \tilde{f}) = df_1 \circ \bar{D}^{*}\bar{D}\tau(\tilde{f}). \tag{3.11}
$$

On the other hand,

$$
\begin{aligned}
R^{M} \;&(d(f_1 \circ \tilde{f})(\tilde{e}_i), \tau(f_1 \circ \tilde{f}))\, d(f_1 \circ f)(\tilde{e}_i) \\
&= R^{f_1^{-1}TM} \;(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f}))\, df_1(d\tilde{f}(\tilde{e}_i)) \\
&= df_1 \circ R^{M} \;(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f}))\, d\tilde{f}(\tilde{e}_i).
\end{aligned} \tag{3.12}
$$

By (3.11) and (3.12), we have

$$
\begin{aligned}
\bar{D}^{*}\bar{D} \;&(f_1 \circ \tilde{f}) + R^{M} \;(d(f_1 \circ \tilde{f})(\tilde{e}_i), \tau(f_1 \circ \tilde{f}))\, d(f_1 \circ \tilde{f})(\tilde{e}_i) \\
&= df_1 \circ \left[\bar{D}^{*}\bar{D}\tau(\tilde{f}) + R^{M} \;(d\tilde{f}(\tilde{e}_i), \tau(\tilde{f}))\, d\tilde{f}(\tilde{e}_i)\right]
\end{aligned} \tag{3.13}
$$

in each $\tilde{U}$. Hence, if f is biharmonic, then $f_1 \circ f$ is also biharmonic. □

Remark 3.5. Theorem 3.4 generalizes the main theorem in [14] into V-manifolds. The condition of f_1 being totally geodesic cannot be weakened into harmonic or biharmonic.

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