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HOMOGENEOUS AND ISOTROPIC COSMOLOGY, THE SCHWARZSCHILD SOLUTION, AND APPLICATIONS

An honors paper submitted to the Department of Mathematics
of the University of Mary Washington
in partial fulfillment of the requirements for Departmental Honors

Pengcheng Zhang
May 2016

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Pengcheng Zhang
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05/07/16

Homogeneous and Isotropic Cosmology, the Schwarzschild Solution, and Applications

Pengcheng Zhang

submitted in partial fulfillment of the requirements for Honors in Mathematics at the University of Mary Washington

Fredericksburg, Virginia

April 2016

Contents

Abstract

Classically, the physics of the universe is described by Newton's Laws of Motion and Newton's Law of Universal Gravitation. In most cases, the results predicted by Newton's theories accurately agree with experimental observations. However, under certain limitations, classical theories may yield slight deviation from observations, such as when the speed of an object approaches the speed of light. At the extreme, classical theory completely fails to explain the motion of photons, which are massless particles of light. In 1915, Albert Einstein published the General Theory of Relativity. Einstein's theory provides a new perspective to a better understanding of the physics describing this universe. In this paper, we attempt to introduce some of the prerequisite material in differential geometry and investigate the general theory of relativity, along with some of its solutions from a mathematical point of view. We study homogeneous and isotropic cosmology, and the Schwarzschild solution. Finally, we will discuss some of their applications and significance.

1 Surfaces and Manifolds

The general theory of relativity is built upon the subject of differential geometry. In this chapter, we shall introduce and discuss some of the main elements of differential geometry (based on [1, 3, 4]) required for the mathematical derivation of general relativity.

1.1 Surfaces

Before we start, we shall establish some notation conventionally used in differential geometry, as some may seem quite different from typical modern notations. Note that the convention in differential geometry is to use superscripts to denote the sequence of terms, as opposed to the ordinary use of subscripts. For example, a point x in n-dimensional Euclidean space $\mathbb{R}^n$ is written as

$$x = (x^1, x^2, ..., x^i, ..., x^n),$$

where x^i is the i-th component of the point x.

We will also spend a great deal of effort discussing vector-valued functions mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$. These functions are often labeled by a bold-faced symbol. For example, a vector-valued function $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ takes an entry $(u^1, u^2) \in \mathbb{R}^2$ and yields a vector $(x^1, x^2, x^3) \in \mathbb{R}^3$, written

$$\mathbf{x}(u^1, u^2) = \left((x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)\right).$$

Most of the functions that we consider are differentiable functions of class C^k, meaning the function can be continuously differentiated k times. Furthermore, we conventionally denote the partial derivatives of a vector-valued function $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ with respect to the i-th variable by means of a subscript, i.e.

$$\frac{\partial \mathbf{x}}{\partial u^i} = \mathbf{x}_i.$$

With the above notation in mind, we start the introduction with the definition of the most basic element of a surface–a simple surface, also known as a coordinate patch. Below is the formal definition of a simple surface.

Definition 1.1.1. Consider a C^k injective function $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ for some $k \geq 1$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$ with coordinates u^1 and u^2. If the cross-product between the two partial derivatives with respect to u^1 and u^2 is nonzero everywhere on the domain $\mathcal{U}$, i.e.

$$\left(\frac{\partial \mathbf{x}}{\partial u^1}\right) \times \left(\frac{\partial \mathbf{x}}{\partial u^2}\right) = \mathbf{x}_1 \times \mathbf{x}_2 \neq \mathbf{0},$$

then we called $\mathbf{x}$ a C^k *simple surface* (or *coordinate patch*).

A C^k simple surface is the basic element of a C^k surface, which we shall define at the end of this section. Below are some examples of C^k simple surfaces.

Example 1.1.2. Let $f(u^1, u^2)$ be a C^k function of two variables mapping from an open set $\mathcal{U}$ of $\mathbb{R}^2$ to $\mathbb{R}$. Let the vector-valued function $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be an injective function of class C^k, defined by

$$\mathbf{x}(u^1, u^2) = (u^1, u^2, f(u^1, u^2)).$$

Note that the cross-product between the two partial derivatives of $\mathbf{x}$ with respect to u^1 and u^2 is

$$\begin{aligned}
\mathbf{x}_1 \times \mathbf{x}_2 &= \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \\
&= \left(1, 0, \frac{\partial f}{\partial u^1}\right) \times \left(0, 1, \frac{\partial f}{\partial u^2}\right) \\
&= (1, 0, f_1) \times (0, 1, f_2) \\
&= (-f_1, -f_2, 1) \neq \mathbf{0}.
\end{aligned}$$

Thus the vector-valued function $\mathbf{x}$ is a C^k simple surface. This function is often referred to as a *Monge patch.*

Example 1.1.3. Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be a function of class C^k defined by

$$\mathbf{x}(u^1, u^2) = \left((u^1)^2, (u^2)^2, u^1 u^2\right).$$

The cross-product between the two partial derivatives of $\mathbf{x}$ with respect to u^1 and u^2 is

$$\begin{aligned}
\mathbf{x}_1 \times \mathbf{x}_2 &= \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \\
&= \left(2u^1, 0, u^2\right) \times \left(0, 2u^2, u^1\right) \\
&= (-2(u^2)^2, -2(u^1)^2, 4u^1 u^2).
\end{aligned}$$

Note that at the point $(u^1, u^2) = (0, 0)$, the cross-product $\mathbf{x}_1 \times \mathbf{x}_2 = (0, 0, 0)$. Thus $\mathbf{x}$ is not a simple surface. Also note that $\mathbf{x}$ is not injective, as

$$\mathbf{x}(1, 1) = \mathbf{x}(-1, -1).$$

A simple fix would be restricting the domain to a subset of $\mathbb{R}^2$ where u^1, u^2 are both positive or negative real number.

Next, we shall introduce the tangent plane and the unit normal of a simple surface.

Definition 1.1.4. Given a simple surface $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$, a point $(a, b) \in \mathcal{U}$, and a point $P \in \mathbb{R}^3$ such that $P = \mathbf{x}(a, b)$, the *tangent plane* to $\mathbf{x}$ at the point P is the plane through P perpendicular to the vector cross-product between the partial derivatives of $\mathbf{x}$ with respect to u^1 and u^2,

$$\left(\frac{\partial \mathbf{x}(a,b)}{\partial u^1}\right) \times \left(\frac{\partial \mathbf{x}(a,b)}{\partial u^2}\right) = \mathbf{x}_1(a,b) \times \mathbf{x}_2(a,b). \tag{1.1.1}$$

The *unit normal* to the surface at P is the directional unit-vector of the above cross-product, defined as

$$\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}, \tag{1.1.2}$$

where the right-hand side is evaluated at the point $P = (a, b)$.

Remark. The tangent plane at the point P is formed by all tangent vectors to the simple surface $\mathbf{x}$ at the point P. Note that the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for the tangent plane. Furthermore, as the unit normal is orthogonal to both partial derivatives, $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{n}\}$ form a basis for $\mathbb{R}^3$.

In order to define a C^k surface, we shall first give a definition for a proper simple surface. Before we do that, let's recall the definition of continuity, specifically, the continuity of a function mapping from a subset of $\mathbb{R}^3$ to $\mathbb{R}^2$.

Definition 1.1.5. Let M be a subset of $\mathbb{R}^3$. A function $f : M \to \mathbb{R}^2$ is *continuous* at a point $P \in M$ if for every open set $\mathcal{O} \subset \mathbb{R}^2$ with $f(P) \in \mathcal{O}$, there is an ϵ-neighborhood $\mathcal{N}$ of P with $f(\mathcal{N}) \subset \mathcal{O}$.

With a clear idea of continuity, we proceed to define a proper simple surface.

Definition 1.1.6. Given a simple surface $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$, if the inverse function $\mathbf{x}^{-1} : \mathbb{R}^3 \to \mathcal{U}$ exists and is continuous, then the simple surface $\mathbf{x}$ is called *proper*.

Definition 1.1.7. Let $\mathcal{V}$ and $\mathcal{U}$ be open subsets of $\mathbb{R}^2$. A C^k *coordinate transformation* is a C^k bijective function $f : \mathcal{V} \to \mathcal{U}$ whose inverse function $f^{-1} : \mathcal{U} \to \mathcal{V}$ is also of class C^k, i.e. both f and g have k continuous derivatives.

Finally, as now we have all the required pieces, we give a formal definition of a surface.

Definition 1.1.8. A C^k *surface* in $\mathbb{R}^3$ is a subset $M \subset \mathbb{R}^3$ such that for every point $P \in M$, there is a proper C^k simple surface whose image is in M and which contains an ϵ-neighborhood of P for some $\epsilon > 0$. Furthermore, if both $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ and $\mathbf{y} : \mathcal{V} \to \mathbb{R}^3$ are such simple surfaces with $\mathcal{U}' = \mathbf{x}(\mathcal{U})$, $\mathcal{V}' = \mathbf{y}(\mathcal{V})$, then $\mathbf{y}^{-1} \circ \mathbf{x} : (\mathbf{x}^{-1}(\mathcal{U}' \cap \mathcal{V}')) \to (\mathbf{y}^{-1}(\mathcal{U}' \cap \mathcal{V}'))$ is a C^k coordinate transformation. (This is described in Figure 1.1.1.)

1.2 Metric Tensor, Christoffel Symbol, and Curvature

So far, we have established the ideas of a simple surface and a C^k surface. Next, we shall investigate some of the operations on simple surfaces. One of the most commonly used operations on vectors is the dot or "inner" product. The dot product takes two vectors and yields a scalar value. It is

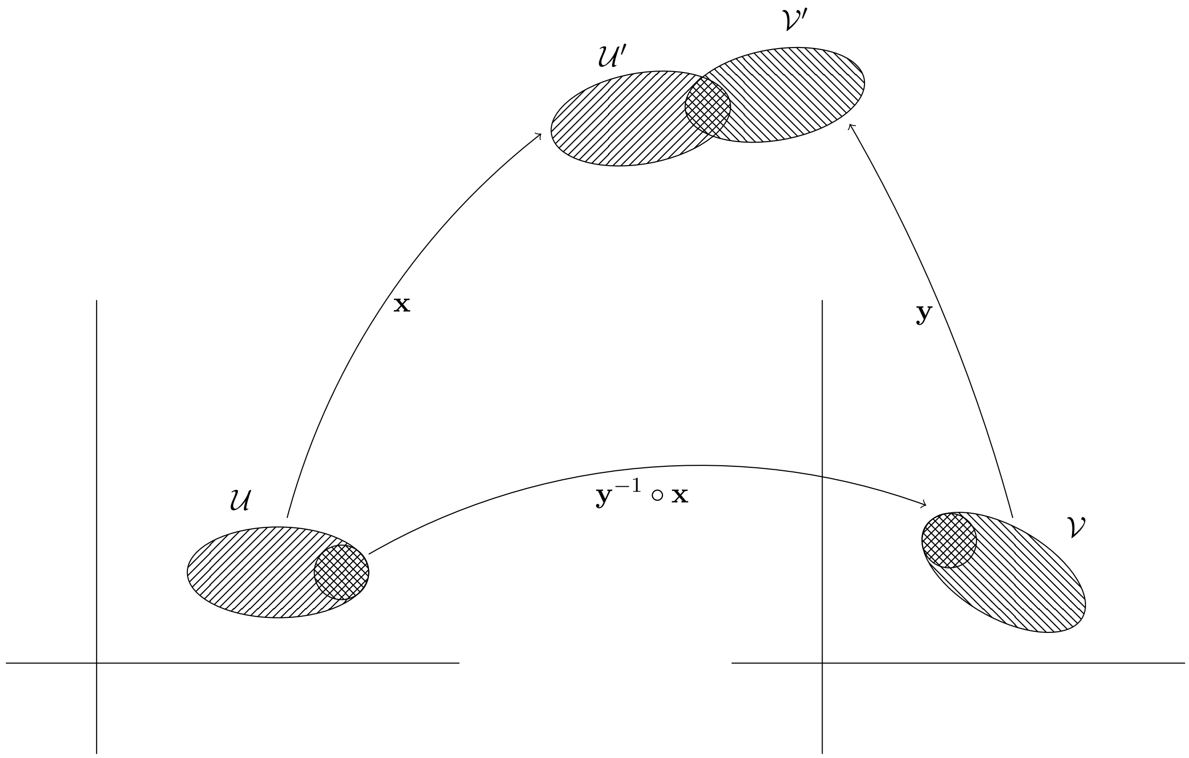

Figure 1.1.1

directly related to the magnitudes of the two vectors and the angle θ between the vectors: given two non-zero vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, the dot product is

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1 \cdot \vec{v}_2 = |\vec{v_1}||\vec{v_2}| \cos \theta.$$

In classical physics, the mechanical work done by a force on an object is defined as the dot product between the force vector and displacement vector. The resultant work gives a sense of how effective the force is in moving the object in the desired direction.

For surfaces, we wish to construct an inner product operation. Consider a surface $M \subset \mathbb{R}^3$, a point $P \in M$, and two vectors $\mathbf{X}$ and $\mathbf{Y}$ tangent to M at P. As we have previously noted, $\mathbf{x}_1$ and $\mathbf{x}_2$ form a basis for the tangent plane to M at point P. Thus all tangent vectors of M at P can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{X} = \sum_i X^i \mathbf{x}_i \quad \text{and} \quad \mathbf{Y} = \sum_j Y^j \mathbf{x}_j,$$

where X^i and Y^j are the i-th and j-th component of the vectors $\mathbf{X}$ and $\mathbf{Y}$, respectively. Furthermore, we can define the inner product, which is a bilinear function, of the two tangent vectors

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i,j} X^i Y^j \langle \mathbf{x}_i, \mathbf{x}_j \rangle.$$

Thus we make the following definition:

$$g_{ij}(u^1,u^2) = \langle \mathbf{x}_i(u^1,u^2), \mathbf{x}_j(u^1,u^2)\rangle \quad \text{or} \quad g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j\rangle. \tag{1.2.1}$$

The inner-product between the vectors $\mathbf{X}$ and $\mathbf{Y}$ can then be written as

$$\langle \mathbf{X}, \mathbf{Y}\rangle = \sum X^i Y^j \langle x_i, y_j\rangle = \sum X^i Y^j g_{ij} \tag{1.2.2}$$

The functions g_{ij} are often referred to as the metric coefficients, the coefficients of the metric tensor, or the coefficients of the Riemannian metric. The coefficients g_{ij} define a matrix

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_1, \mathbf{x}_1\rangle & \langle \mathbf{x}_1, \mathbf{x}_2\rangle \\ \langle \mathbf{x}_2, \mathbf{x}_1\rangle & \langle \mathbf{x}_2, \mathbf{x}_2\rangle \end{pmatrix}.$$

Note that the commutative nature of the inner product operation establishes a symmetry on the matrix formed by the coefficients g_{ij}, i.e. $g_{12} = g_{21}$. We shall also introduce some new notations regarding these metric coefficients:

$$g = \det(g_{ij}) = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}g_{22} - 2g_{12}, \tag{1.2.3}$$

$$g^{kl} = (k,l) \text{ entry of the inverse matrix of } (g_{ij}). \tag{1.2.4}$$

The inverse matrix of (g_{ij}) is given by

$$(g^{ij}) = (g_{ij})^{-1} = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix}. \tag{1.2.5}$$

With the coefficients of the metric tensor defined, we can proceed to construct the *Christoffel symbols*.

Definition 1.2.1. Given a simple surface $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ on the surface M where $\mathcal{U} \subset \mathbb{R}^2$, the *Christoffel symbols* are functions Γ^l_{ij} $(1 \le i,j,l \le 2)$ defined on $\mathcal{U}$ by

$$\Gamma^l_{ij} = \sum_{n=1}^{2} \langle \mathbf{x}_{ij}, \mathbf{x}_n\rangle g^{nl} = \frac{1}{2}\sum_{k=1}^{2} g^{kl}\left(\frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{kj}}{\partial u^i}\right). \tag{1.2.6}$$

Note that the Christoffel symbols can be completely defined by the metric coefficients. Due to this property, we say that Γ^l_{ij} is *intrinsic*.

Next we define the Riemannian curvature tensor.

Definition 1.2.2. Given a simple surface $\mathbf{x} : \mathcal{U} \to \mathbb{R}^3$ on the surface M where $\mathcal{U} \subset \mathbb{R}^2$, the *Riemannian curvature tensor* with index (i,j,k,l) is defined by

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \sum_{p=1}^{2}\left(\Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}\right) \tag{1.2.7}$$

for all $1 \le i,j,k,l \le 2$.

Since the Christoffel sysbols are intrinsic and the Riemannian curvature tensor is completely defined by Γ^l_{ij}, the Riemannian curvature tensor is also intrinsic.

1.3 Manifolds

Thus far, we have established the concepts of surfaces and some operations regarding simple surfaces. A surface is a 2-dimensional manifold in $\mathbb{R}^3$ that exhibits the structure of $\mathbb{R}^2$ locally. In general, an n-dimensional manifold will exhibit the structure of $\mathbb{R}^n$ locally.

Definition 1.3.1. Let M be a metric space and p some point in M. A *n-dimensional coordinate chart about the point p* is a neighborhood $\mathcal{N}$ of p and a continuous, injective function $\phi : \mathcal{N} \to \mathbb{R}^n$ such that the image of $\mathcal{N}$ under ϕ is open in $\mathbb{R}^n$. If $\phi^{-1} : \phi(\mathcal{N}) \to \mathcal{N} \subset M$ is continuous, then$(\mathcal{N}, \phi)$ is a *proper coordinate chart*.

Recall the definition of a coordinate transformation in $\mathbb{R}^2$ (Definition 1.1.7). Expanding the definition to $\mathbb{R}^n$ givens us the idea of a C^k diffeomorphism.

Definition 1.3.2. Let $\mathcal{U}$ and $\mathcal{V}$ be open sets in $\mathbb{R}^n$ and the function $f : \mathcal{U} \to \mathcal{V}$ be C^k. If the inverse function $f^{-1} : \mathcal{V} \to \mathcal{U}$ is also of class C^k, then f is called a *C^k diffeomorphism*.

Generally speaking, a manifold M is a topological space whose elements are labeled by elements of $\mathbb{R}^n$, i.e. a point $p \in M$ is labeled by a point $x^k \in \mathbb{R}^k$.

Definition 1.3.3. Let M be a metric space. The space M is an *n-dimensional C^∞ manifold* if there is a collection $\mathcal{A}$ of coordinate charts $(\mathcal{U}, \phi)$, called the *atlas* of M, such that

1. for each $p \in M$, there exists a n-dimensional proper coordinate chart $(\mathcal{U}, \phi) \in \mathcal{A}$ where $p \in \mathcal{U}$;
2. if $(\mathcal{U}, \phi), (\mathcal{V}, \psi) \in \mathcal{A}$ and $\mathcal{U}$ and $\mathcal{V}$ are non-disjoint, i.e. $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, then
$$\psi \circ \phi^{-1} : \phi(\mathcal{U} \cap \mathcal{V}) \to \psi(\mathcal{U} \cap \mathcal{V})$$
is a C^∞ diffeomorphism; and
3. $\mathcal{A}$ contains all possible charts with properties 1 and 2. (See Figure 1.3.1.)

1.4 Connections on Manifolds

Now, we shall formally define a way to differentiate a vector field. Let (M, g) be a smooth manifold with a smooth metric g. First, we make use of some set notation.

Given a manifold M, we let $\mathscr{F}(M)$ denote the set of all infinitely differentiable functions from M to $\mathbb{R}$, i.e. $\mathscr{F}(M) = \{f : M \to \mathbb{R} | f \text{ is of class } C^\infty\}$. Let $\mathfrak{X}(M)$ denote the set of all vector fields on M.

Definition 1.4.1. If $X, Y \in \mathfrak{X}(M)$, then the *Lie bracket* of X and Y, $[X, Y]$, is the field of vectors defined by
$$[X, Y]_p f = X_p(Yf) - Y_p(Xf) \quad \text{for } f \in \mathscr{F}(M) \text{ and } p \in M. \tag{1.4.1}$$
Note that $[X, Y]$ is a vector field.

Definition 1.4.2. Given a manifold M, a *connection* on M with respect to a metric g is a function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\nabla(X, Y)$ is denoted $\nabla_X Y$, such that for $X, Y, Z \in \mathfrak{X}(M)$, $r \in \mathbb{R}$ and $f \in \mathscr{F}(M)$, the following conditions are satisfied:

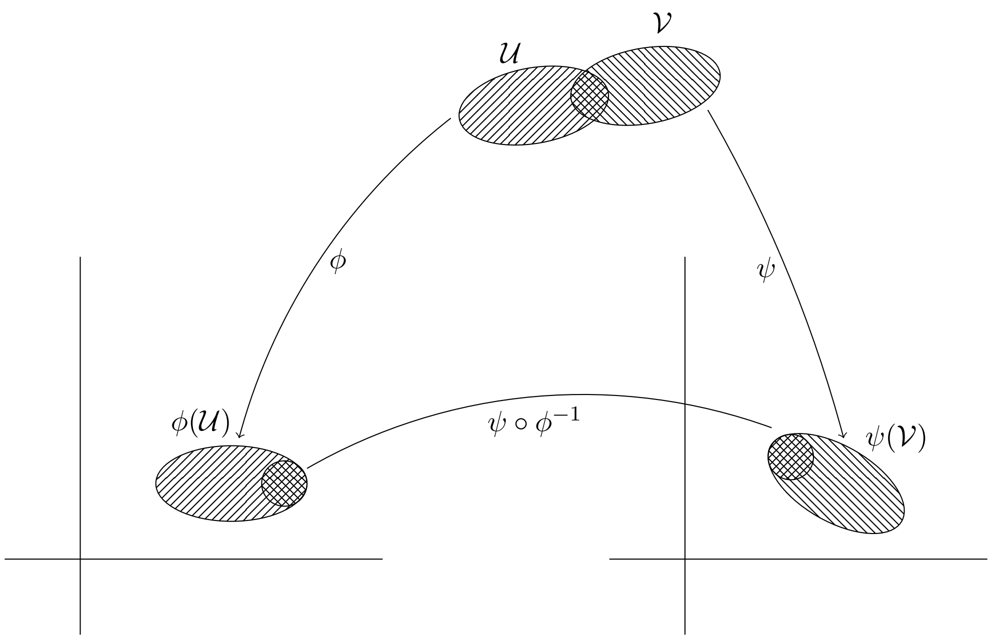

Figure 1.3.1

1. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$ and $\nabla_X rY = r\nabla_X Y$;
2. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$ and $\nabla_{fX} Y = f\nabla_X Y$; and
3. $\nabla_X fY = (Xf)Y + f\nabla_X Y$.

Here, ∇ is called a *connection with respect to* g, and $\nabla_X Y$ is the *covariant derivative* of Y in the direction of X.

Definition 1.4.3. Given a manifold M, the *Riemann-Christoffel curvature tensor of type* (1,3) is the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]Z}.$$

In local coordinates,

$$R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^i} = \sum R^l_{ijk} \frac{\partial}{\partial x^l},\ 1 \leq i,j,k,l \leq n,$$

where x^i are the local coordinates in M.

Lastly, we give a definition of the Ricci curvature tensor.

Definition 1.4.4. Let T_pM denote the tangent space of the n-dimensional manifold M at a point p. For any pair of tangent vectors X_p and Y_p at the point p, we consider a function $\Xi_p(X_p, Y_p) : T_pM \to T_pM$ defined by

$$\Xi_p(X_p, Y_p)V_p = R(V_p, X_p)Y_p.$$

Note that $\Xi_p(X_p, Y_p)$ is actually a linear transformation for every point p and vector pair $X_p, Y_p \in T_pM$. The *Ricci curvature tensor* evaluated at each point $p \in M$ is a function $S_p : T_pM \times T_pM \to \mathbb{R}$ defined by

$$S_p(X_p, Y_p) = \mathrm{trace}(\Xi_p(X_p, Y_p)).$$

In local coordinates, we have

$$S_p = R_{ij} dx^i \otimes dx^j,$$

where $R_{ij} = R^k_{ikj}$. In terms of the Riemann curvature tensor and the Christoffel symbols, we have

$$R_{ij} = R^k_{ikj} = \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}.$$

The tensor R_{ij} is called a *Ricci tensor.*

2 Special Relativity and General Relativity

2.1 Introduction

In this chapter, we shall investigate the physics of this universe using the mathematics that we have defined in the previous chapter, and construct the core ideas for the general theory of relativity based on [2, 5, 6].

In pre-relativity time, it was assumed that space takes the manifold structure of $\mathbb{R}^3$. The addition of a time dimension gives a 4-dimensional spacetime taking the manifold structure of $\mathbb{R}^4$. The movement of an object in spacetime is described by Newtonian mechanics and Newton's theory of gravitation: Newton's three laws of motion and Newton's law of universal gravitation. Newton says that any object with mass will induce a gravitational force on other objects in space. The gravitational attraction between two objects has the magnitude

$$F_g = \frac{Gm_1m_2}{r^2}, \tag{2.1.1}$$

where G is the universal gravitational constant, m_1 and m_2 are the inertial masses of the two objects, and r is the metric distance between the two objects. In most cases, Newton's theory is astoundingly accurate and sufficient. We need no more than Newton's theory to send a rocket to the moon and back to earth safely. However, Newton's theory fails when the speed of the object approaches the speed of light. Furthermore, Newton's theory has some fundamental flaws. There are three main problems with Newton's theory of gravitation:

1. Newton's theory says that the strength of the gravitational force is proportional to the inertial masses of the two objects, much like Coulomb's law for the electrical force between two charged particles. However, no one, including Newton, can explain why that's the case. The source, and carrier, for this gravitational force is unknown.

2. Newton's theory describes this gravitational force to be instantaneous. That is, if the sun were to instantaneously disappear, the earth will immediately fall out of the orbit around the sun irrespective of the distance between the sun and earth. This means that gravity travels faster than light! (It takes the light from the sun roughly 8 minutes and 20 seconds to reach the surface of earth.) But how?

3. Experimental results contradict Newton's theory in some cases. Newton's theory predicts Mercury's orbit around the sun to be a constant, elliptical orbit, yet the experimental observations showed that Mercury's orbit is constantly rotating. Also, Newton's theory suggests that gravity has no effect on light, since photons are massless. This means that we should not see any stars "behind" the sun, as the sun is in the way and light only travels in a straight line. This is also proven wrong, as we are actually able to observe stars which Newton's theory predicts we weren't supposed to see – gravity can bend light!

We will discuss the details of special relativity and general relativity, and the derivation of Einstein's equation in the following sections.

2.2 Special Relativity

Imagine a person standing still on a car moving with a constant velocity. To an observer some distance away, the person, along with the car, is moving at a constant velocity. But notice that to the person standing on the car, he is stationary as well as the car, what's moving is the surroundings. The motion of an object is dependent on the frame of reference. Now, suppose that the person has a laser pointer, which he points in the direction that the car is moving towards and shoots the laser beam. To the person holding the laser pointer, the laser beam is traveling at the speed of light moving away from him. In classical physics, we would then think that, in the reference frame of the observer, the laser pointer is traveling at a constant velocity as is the car. The laser beam is leaving the tip of the laser pointer at the speed of light in the same direction as the velocity of the laser pointer. Thus to the observer, it is logical to think that the laser beam is in fact traveling at a speed of the sum of the speed of light and the speed of the car. However, this is not the case. Experimental results, with all possible errors considered, show that the laser will travel in exactly the speed of light regardless of the reference frame chosen. That is, the laser will travel at the speed of light in the eyes of the person holding the laser, yet also the same speed in the eyes of the observer regardless of where he chooses to stand in space. The theory that fixes this problem is the special theory of relativity.

In special relativity, spacetime is assumed to have the manifold structure of $\mathbb{R}^4$. Furthermore, special relativity regards only inertial motions in spacetime, or nonaccelerating motions. In the reference frame of an inertial observer, events in spacetime are labeled by three spatial coordinates x^1, x^2, and x^3, and a time coordinate $t = x^0$. This map of spacetime into $\mathbb{R}^4$ is called a *global inertial coordinate system*. In ordinary Euclidean space, the distance between two points $x, y \in \mathbb{R}^3$ is an intrinsic property and is invariant under coordinate transformations, defined as

$$D^2 = (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2.$$

In special relativity, the metric for the 4-dimensional spacetime requires a slightly different consideration. We want to define a spacetime interval that is independent of the observer's frame of reference. The spacetime interval I between two events x and y is defined by

$$I = -(x^0 - y^0)^2 + \frac{1}{c^2}\left[(x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2\right], \tag{2.2.1}$$

where c is the speed of light. Changing the units of the speed of light such that $c = 1$ yields

$$I = -(x^0 - y^0)^2 + (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2. \tag{2.2.2}$$

We then define the metric of spacetime dI by

$$dI = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{2.2.3}$$

In index notation, we define the metric of spacetime η_{ab}

$$\eta_{ab} = \sum_{i,j=0}^{3} \eta_{ij}(dx^i)_a(dx^j)_b, \tag{2.2.4}$$

where x^i is any coordinate of the global coordinate system and η_{ij} is an entry of the matrix

$$(\eta_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similar to the distance metric D^2 in Euclidean space, the tensor field η_{ab} is independent of the choice of global inertial coordinate system.

Let's consider the case where two inertial observers, O and O', are present in space. Observer O labels an event p with spacetime coordinates (t, x, y, z). Observer O' moves with velocity v in the x-direction, passing observer O at the event labeled by $t = x = y = z = 0$. The two observers have synchronized clocks. Thus at the instant that the two observers pass each other, the two frames of reference overlap. In classical Newtonian mechanics, event p will be labeled by observer O' by the following coordinates

$$t' = t, \tag{2.2.5}$$
$$x' = x - vt, \tag{2.2.6}$$
$$y' = y, \tag{2.2.7}$$
$$z' = z. \tag{2.2.8}$$

Note that this is simply a linear coordinate transformation from reference frame O to reference frame O' with respect to time. In special relativity, however, the labeling of event p by observer O' will be related to the labeling of observer O by a Lorentz transformation, given by

$$t' = \frac{(t - vx/c^2)}{\sqrt{1 - v^2/c^2}}, \tag{2.2.9}$$
$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{2.2.10}$$
$$y' = y, \tag{2.2.11}$$
$$z' = z, \tag{2.2.12}$$

where c is the speed of light. Notice that, although the two observers synchronized their clocks at an instant of time ($t = 0$), observer O' experiences time differently than O. Note that differentiating t' with respect to t yields

$$\frac{\partial t'}{\partial t} = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{2.2.13}$$

since v is constant and invariant of time. In this case, if $v = 0$, then the two clocks run at the same speed. However, if $v > 0$, then $dt' < dt$, meaning the clock of O' runs slower than the clock of O. This phenomenon is referred to as time dilation–the faster an O' moves, the slower the time is, relativistically. The same concept can be applied to spatial coordinates, such as x and x', yielding the idea of length contraction–the faster the object moves, the shorter the displacement is, relativistically.

The special theory of relativity improves upon classical Newtonian mechanics. However, the fundamental flaws of Newtonian mechanics remain in special relativity, one of which being the inperfect concept of gravity. To truly fix the flaws of classical physics, we must reconsider the basic assumptions of physics, as Einstein did.

2.3 General Relativity

Although Newtonian mechanics and Newton's theory of gravitation predict the motion of objects in this universe quite accurately in most cases, the flaws in the system are undeniable. Even Einstein could not find an explanation for the mysterious force of gravitation, so he chose to start from scratch and devise a new theory that can explain the "gravitational pull" between two objects. The result is the theory of general relativity.

In general relativity, the spacetime metric is not flat, as it was assumed in classical mechanics and special relativity. The motion of an object is not affected by the existence of another object. In other words, there is no such force as gravitation. Instead, the existence of an object curves the spacetime around it. Imagine the universe being a large, initially flat, trampoline in a vacuum with no gravitation. A small object, say the moon, on the trampoline will remain at rest unless a force is applied. Now imagine a large object, say the earth, being placed at some distance away from the moon on the trampoline, forcing the surface of the trampoline to curve around the earth. Still, no force is applied on the moon. However, it is now resting on a curved trampoline with a valley some distance away. Naturally, earth starts to move toward the low point of the valley. Note that earth itself curves the trampoline around it as well, however slightly compared to the sun. This causes the sun to naturally move toward the earth, however slightly. This overlaps perfectly with the Newtonian image of gravitational pull between two objects in space. Now that we have a basic image of the core idea of general relativity, let's proceed to define the theory mathematically.

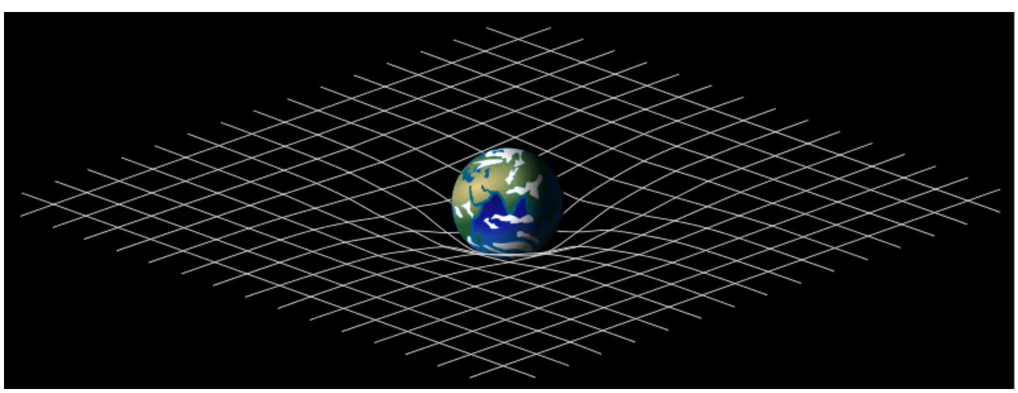

Figure 2.3.1: Earth curves the spacetime metric around it. By Mysid (Own work. Self -made in Blender & Inkscape.) [CC BY-SA 3.0 (http://creativecommons.org/licenses/by-sa/3.0) or GFDL (http://www.gnu.org/copyleft/fdl.html)], via Wikimedia Commons.

Let spacetime be a manifold M defined by the metric g_{ij}. We shall construct general relativity in a way such that the curvature of g_{ij} is related to the distribution of matter in spacetime, i.e. curvature is correlated with mass. The two basic principles that govern the laws of physics in general relativity are:

1. the principle of general covariance states that the metric g_{ij} and quantities derived from the metric are the only spacetime terms that can appear in any equations of physics, i.e. all equations of physics must be intrinsic;

2. when taking g_{ij} to be the flat metric η_{ij}, all equations of physics must reduce down to those in special relativity.

Relativity states that material particles in spacetime take paths that are time-like curves. Simply put, this means that particles travel along the axis of time. Time-like curves can be parametrized by the *proper time* τ, defined as

$$\tau = \int (-g_{ab}T^a T^b)^{1/2}\ dt, \tag{2.3.1}$$

where t is an arbitrary parametrization of the time-like curve, and T^a is the tangent to the curve in this parameterization. We then define the 4-velocity of a particle with respect to the proper time to be the unit tangent to its world line (geodesic),

$$u^a = \frac{\partial x^a}{\partial \tau}.$$

Thus the 4-momentum of the particle is defined by

$$p^a = mu^a. \tag{2.3.2}$$

Given an observer in spacetime with the 4-velocity v^a, the energy of the particle measured by the observer is

$$E = -p_a v^a. \tag{2.3.3}$$

In general relativity, continuous matter distributions and fields are described by a stress-energy tensor T_{ab}. For a *perfect fluid*, or a continuous matter distribution, the stress-energy tensor is given as

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b), \tag{2.3.4}$$

where ρ is the mass-energy density of the matter, and P is the pressure of the fluid as measured in its rest frame. This stress-energy tensor satisfies the equation of motion

$$\nabla^a T_{ab} = 0, \tag{2.3.5}$$

yielding

$$u^a \nabla_a \rho + (\rho + P)\nabla^a u_a = 0\ , \tag{2.3.6}$$
$$(P + \rho)u^a \nabla_a u_b + (g_{ab} + u_a u_b)\nabla^a P = 0\ . \tag{2.3.7}$$

The underlying idea behind the general theory of relativity is that the presence of matter curves the metric around it. Viewing the stress-energy tensor as the equivalence of mass in classical physics, we seek a model that relates the stress-energy tensor with the metric tensor of spacetime. The resulting model is known as Einstein's field equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}. \tag{2.3.8}$$

First note that this equation is completely defined by the metric tensor g_{ij}, satisfying the first starting principle for general relativity. On the other hand, when the metric is replaced with η_{ij}, the result should be consistent with that of special relativity. Upon examining several physical relationships, it turns out that Einstein's field equation is in agreement with special relativity. We will take a closer look at this equation in the following section.

2.4 Einstein's Equation

To derive Einstein's field equation, we start with the Hilbert action:

$$S = \int \sqrt{-g}d^4x \left(\frac{c^4}{16\pi G}R + \mathcal{L}_{\mathcal{M}}\right), \tag{2.4.1}$$

where R is the Ricci scalar and $\mathcal{L}_{\mathcal{M}}$ is the field of matter distribution. The Hilbert action must follow the equation of motion (also know as the action principle), where any variation in S must equal zero. Thus we simplify the variation in the Hilbert action:

$$\begin{aligned}
\delta S &= \delta \left[\int \sqrt{-g}d^4x \left(\frac{c^4}{16\pi G}R + \mathcal{L}_{\mathcal{M}}\right)\right] \\
&= \delta \left[\int d^4x \left(\frac{c^4}{16\pi G}\sqrt{-g}R + \sqrt{-g}\mathcal{L}_{\mathcal{M}}\right)\right] \\
&= \int d^4x \left(\frac{c^4}{16\pi G}\delta\left(\sqrt{-g}R\right) + \delta\left(\sqrt{-g}\mathcal{L}_{\mathcal{M}}\right)\right) \\
&= \int d^4x \left(\frac{c^4}{16\pi G}\left(\sqrt{-g}\delta R + R\delta\sqrt{-g}\right) + \delta\left(\sqrt{-g}\mathcal{L}_{\mathcal{M}}\right)\right) \\
&= \int d^4x \left(\frac{c^4}{16\pi G}\left(\sqrt{-g}\frac{\delta R}{\delta g^{\mu\nu}} + R\frac{\delta\sqrt{-g}}{g^{\mu\nu}}\right) + \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathcal{M}}\right)}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu} \\
&= \int \sqrt{-g}d^4x \left(\frac{c^4}{16\pi G}\left(\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{g^{\mu\nu}}\right) + \frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathcal{M}}\right)}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu}.
\end{aligned} \tag{2.4.2}$$

Note that the Ricci scalar can be written in terms of the Ricci tensor

$$R = R_\mu^{\ \mu} = g^{\mu\nu}R_{\mu\nu}.$$

Thus we have

$$\begin{aligned}
\frac{\delta R}{\delta g^{\mu\nu}} &= \frac{\delta(g^{\mu\nu}R_{\mu\nu})}{\delta g^{\mu\nu}} \\
&= R_{\mu\nu}\frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}} + g^{\mu\nu}\frac{\delta R_{\mu\nu}}{\delta g^{\mu\nu}} \\
&= R_{\mu\nu} + g^{\mu\nu}\frac{\delta R_{\mu\nu}}{\delta g^{\mu\nu}}.
\end{aligned}$$

The second term in the expression above is a total derivative, which will not contribute to the variation of the function. Thus we ignore this term for now. Now we turn our attention back to the second term in Equation 2.4.2:

$$\begin{aligned}\frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{g^{\mu\nu}} &= \frac{R}{\sqrt{-g}}\frac{1}{2}\sqrt{-g}g^{\mu\nu}\frac{\delta g_{\mu\nu}}{\delta g^{\mu\nu}} \\ &= \frac{R}{2}g^{\mu\nu}\frac{\delta g_{\mu\nu}}{\delta g^{\mu\nu}} \\ &= -\frac{R}{2}g_{\mu\nu}\frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}} \\ &= -\frac{1}{2}g_{\mu\nu}R.\end{aligned}$$

We then define the stress-energy tensor

$$\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathcal{M}})}{\delta g^{\mu\nu}} = -\frac{1}{2}T_{\mu\nu}.$$

Thus we can rewrite Equation 2.4.2 as

$$\begin{aligned}\delta S &= \int\sqrt{-g}d^4x\left(\frac{c^4}{16\pi G}\left(\frac{\delta R}{\delta g^{\mu\nu}}+\frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{g^{\mu\nu}}\right)+\frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathcal{M}}\right)}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu} \\ &= \int\sqrt{-g}d^4x\left(\frac{c^4}{16\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)-\frac{1}{2}T_{\mu\nu}\right)\delta g^{\mu\nu}.\end{aligned} \tag{2.4.3}$$

Setting the integrand to zero would yield an extremum for the variation of the Hilbert action:

$$\begin{aligned}\frac{c^4}{16\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)-\frac{1}{2}T_{\mu\nu}&=0 \\ \frac{c^4}{16\pi G}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)&=\frac{1}{2}T_{\mu\nu} \\ R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R&=\frac{8\pi G}{c^4}T_{\mu\nu}.\end{aligned} \tag{2.4.4}$$

Finally, setting the units of G and c such that the magnitudes of both constants are 1 will yield

$$G_{\mu\nu} = R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \tag{2.4.5}$$

which is Einstein's field equation. To check the agreement between general relativity and special relativity, we replace the metric tensor $g_{\mu\nu}$ with the flat Minkowski metric $\eta_{\mu\nu}$ and check the resulting physical laws.

In short, general relativity states that spacetime is a manifold M with a metric g_{ab}. The curvature of this metric is related to the matter distribution in spacetime, described by Einstein's field equation. Mathematically, Einstein's field equation can be viewed as a coupled system of nonlinear second-order partial differential equations for the metric g_{ab}. The difficulty in the application of this wonderful equation is the need to solve for the spacetime metric and the matter distribution simultaneously. If we are able to obtain all exact solutions to Einstein's field equation, then we can accurately describe the past history, explain current events, and even predict the future of this universe using the laws of physics as we currently know them.

3 Cosmology, Schwarzschild Solution and Applications

In the previous chapter, we have discussed and derived Einstein's field equation, which related the spacetime metric g_{ab} with matter distribution. This invisible, intangible metric of spacetime governs the laws of physics in our universe. Using the general theory of relativity, we are now capable of calculating the origin, evolution, and eventual fate of the universe, under the conditions that the distribution of matter in the universe and the metric of our spacetime manifold is given. The local matter distribution (near our solar system) can be approximated by conducting experiments. After all, this is how we collect data and predict the trajectory required to safely send a rocket to the space. However, unlike rocket science under classical Newtonian mechanics which assumes a spacetime with a flat Euclidean metric, we do not know the actual metric that governs the spacetime of this universe. Furthermore, information about the matter distribution outside of Earth's proximity can be extremely hard to come by and experimentally confirm. Despite Einstein's genius and heroic effort for formulating the general theory of relativity, the subject of cosmology goes as far as our ability for solving Einstein's field equation for the spacetime metric and matter distribution simultaneously.

In this chapter, we will look at some of the known solutions to Einstein's field equations, namely the Robertson-Walker model and the Schwarzschild metric.

3.1 Homogeneous and Isotropic Cosmology

The homogeneity of space refers to the spatial symmetry of points in space. On the other hand, a space being isotropic means that there are no preferred directions in space. It is then natural to assume that our universe is homogeneous and isotropic. The resulting four-dimensional spacetime metric g_{ab} of this isotropic, homogeneous universe is given by

$$g_{ab} = -u_a u_b + h_{ab}(t), \tag{3.1.1}$$

where for each t, $h_{ab}(t)$ is the metric of either a sphere, the flat Euclidean space, or a hyperboloid. Expressing the coordinates with respect to proper time τ, the spacetime metric takes the form

$$ds^2 = -d\tau^2 + a^2(\tau)\begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) & \text{(Spherical)} \\ dx^2 + dy^2 + dz^2 & \text{(Euclidean)} \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2)\ , & \text{(Hyperboloid)} \end{cases} \tag{3.1.2}$$

where $a(\tau)$ is a function of proper time. The general form of this metric is known as the *Robertson-Walker* cosmological model. We can then apply this metric to Einstein's field equation in order to derive laws of physics.

Given Einstein's equation

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab},$$

the stress-energy tensor of the general perfect fluid

$$T_{ab} = \rho u_a u_b + P(g_{ab} + n_a u_b),$$

and that

$$G_{\tau\tau} = G_{ab}u^a u^b \quad \text{and} \quad G_{**} = G_{ab}s^a s^b,$$

where s^a is any unit vector tangent to the homogeneous hypersurfaces, we can attempt to solve for the spatial geometry of the universe under certain constraints. Consider the flat Euclidean spatial geometry,

$$ds^2 = -d\tau^2 + a^2(\tau)(dx^2 + dy^2 + dz^2). \tag{3.1.3}$$

First note that

$$G_{\tau\tau} = 8\pi T_{\tau\tau} = 8\pi\rho, \tag{3.1.4}$$
$$G_{**} = 8\pi T_{**} = 8\pi P. \tag{3.1.5}$$

Recall the computation of the Christoffel symbol,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\sum_{\sigma} g^{\rho\sigma}\left\{\frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right\}.$$

Thus we have the nonvanishing components of the Christoffel symbol

$$\begin{aligned}\Gamma^{\tau}_{xx} = \Gamma^{\tau}_{yy} = \Gamma^{\tau}_{zz} &= \frac{1}{2}\sum_{\sigma} g^{\tau\sigma}\left\{\frac{\partial g_{x\sigma}}{\partial x^{x}} + \frac{\partial g_{x\sigma}}{\partial x^{x}} - \frac{\partial g_{\mu x}}{\partial x^{\sigma}}\right\} \\ &= \frac{1}{2}\sum_{\sigma} g^{\tau\sigma}\left\{2\frac{\partial g_{x\sigma}}{\partial x^{x}} - \frac{\partial g_{\mu x}}{\partial x^{\sigma}}\right\} \\ &= a\dot{a} \qquad (3.1.6)\\ \Gamma^{x}_{x\tau} = \Gamma^{x}_{\tau x} = \Gamma^{y}_{y\tau} = \Gamma^{y}_{\tau y} = \Gamma^{z}_{z\tau} = \Gamma^{z}_{\tau z} &= \frac{1}{2}\sum_{\sigma} g^{x\sigma}\left\{\frac{\partial g_{\tau\sigma}}{\partial x^{x}} + \frac{\partial g_{x\sigma}}{\partial x^{\tau}} - \frac{\partial g_{x\tau}}{\partial x^{\sigma}}\right\} \\ &= \frac{\dot{a}}{a}, \qquad (3.1.7)\end{aligned}$$

where $\dot{a} = da/d\tau$. Thus we have the independent Ricci tensor components

$$\begin{aligned}R_{\tau\tau} &= \sum_{\nu}\frac{d}{dx^{\nu}}\Gamma^{\nu}_{\tau\tau} - \frac{d}{dx^{\tau}}\left(\sum_{\nu}\Gamma^{\nu}_{\nu\tau}\right) + \sum_{\alpha,\nu}\left(\Gamma^{\alpha}_{\tau\tau}\Gamma^{\nu}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\tau}\Gamma^{\nu}_{\alpha\tau}\right) \\ &= -3\ddot{a}/a, \qquad (3.1.8)\\ R_{**} &= \sum_{\nu}\frac{d}{dx^{\nu}}\Gamma^{\nu}_{**} - \frac{d}{dx^{*}}\left(\sum_{\nu}\Gamma^{\nu}_{\nu*}\right) + \sum_{\alpha,\nu}\left(\Gamma^{\alpha}_{**}\Gamma^{\nu}_{\alpha\nu} - \Gamma^{\alpha}_{\nu*}\Gamma^{\nu}_{\alpha*}\right) \\ &= a^{-2}R_{xx} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}. \qquad (3.1.9)\end{aligned}$$

Thus we have

$$R = -R_{\tau\tau} + 3R_{**} = 3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{3.1.10}$$

Thus we obtain

$$\begin{aligned}
G_{\tau\tau} &= R_{\tau\tau} - \frac{1}{2}Rg_{\tau\tau} \\
&= R_{\tau\tau} + \frac{1}{2}R \\
&= -3\ddot{a}/a + 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \\
&= 3\frac{\dot{a}^2}{a^2} = 8\pi\rho\ , \qquad (3.1.11) \\
G_{**} &= R_{**} - \frac{1}{2}Rg_{**} \\
&= R_{**} - \frac{1}{2}R \\
&= \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \\
&= -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = 8\pi P\ . \qquad (3.1.12)
\end{aligned}$$

Furthermore, we can rearrange and combine Equations 3.1.11 and 3.1.12 and obtain

$$\begin{aligned}
-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} &= 8\pi P \\
-6\frac{\ddot{a}}{a} - 3\frac{\dot{a}^2}{a^2} &= 24\pi P \\
-6\frac{\ddot{a}}{a} - 8\pi\rho &= 24\pi P \\
-6\frac{\ddot{a}}{a} &= 8\pi\rho + 24\pi P \\
3\frac{\ddot{a}}{a} &= -4\pi\left(\rho + 3P\right). \qquad (3.1.13)
\end{aligned}$$

For spherical and hyperboloid, we have

$$ds^2 = -d\tau^2 + a^2(\tau)(d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)), \qquad (3.1.14)$$

$$ds^2 = -d\tau^2 + a^2(\tau)(d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2)). \qquad (3.1.15)$$

Applying the same procedure as above, we obtain that

$$\frac{3\dot{a}}{a^2} = 8\pi\rho - 3k/a^2, \qquad (3.1.16)$$

$$\frac{3\ddot{a}}{a} = -4\pi(\rho + 3P). \qquad (3.1.17)$$

where $k = +1$ for the 3-sphere, $k = 0$ for flat Euclidean space, and $k = -1$ for a hyperboloid. Equations 3.1.16 and 3.1.17 are known as the general evolution equations for homogeneous, isotropic cosmology.

3.2 Robertson-Walker Cosmologies

First consider the general evolution equations for homogeneous, isotropic cosmology:

$$3\dot{a}/a^2 = 8\pi\rho - 3k/a^2, \tag{3.2.1}$$

$$3\ddot{a}/a = -4\pi(\rho + 3P). \tag{3.2.2}$$

The k term is related to the spatial geometry, where $k = +1$ for the 3-sphere, $k = 0$ for flat Euclidean space, and $k = -1$ for hyperboloid. We also need to determine the pressure of the system. When the pressure is zero, there is no collision between particles. On the cosmic scale, each of the galaxies can be idealized as a "grain of dust," referring to particles with no collision. However, a thermal distribution of radiation at a temperature of about 3 Kelvin fills the universe. This means that the pressure is nonzero. For massless thermal radiation, the pressure is $P = \rho/3$. Thus the two cases we are concerned about are $P = 0$ and $P = 3$.

Multiplying equation 3.2.1 by a^2 yields

$$3\dot{a} = 8\pi\rho a^2 - 3k. \tag{3.2.3}$$

We then differentiate with respect to τ,

$$3\ddot{a} = 16\pi\rho a\dot{a} + 8\pi\dot{\rho}a^2, \tag{3.2.4}$$

$$\frac{3\ddot{a}}{a} = 16\pi\rho\dot{a} + 8\pi\dot{\rho}a. \tag{3.2.5}$$

Combining with equation 3.2.2 yields

$$\dot{\rho} + 3(\rho + P)\frac{\dot{a}}{a} = 0. \tag{3.2.6}$$

Note that for dust ($P = 0$), we find

$$\rho a^3 = \text{constant}, \tag{3.2.7}$$

while for radiation ($P = \rho/3$), we find

$$\rho a^4 = \text{constant}. \tag{3.2.8}$$

Thus we obtain, for dust,

$$\dot{a}^2 - C/a + k = 0, \tag{3.2.9}$$

where $C = 8\pi\rho a^3/3$ is constant; and for radiation,

$$\dot{a}^2 - C'/a^2 + k = 0, \tag{3.2.10}$$

where $C' = 8\pi\rho a^4/3$. For radiation, first consider the 3-sphere. Taking $k = 1$, we have

$$\dot{a}^2 - C'/a^2 + 1 = 0. \tag{3.2.11}$$

Solving the differential equation yields

$$\begin{aligned} a(\tau) &= \sqrt{-\tau^2 + 2\tau\sqrt{C'}} \\ &= \sqrt{C'}\sqrt{-\tau^2/C' + 2\tau/\sqrt{C'}} \\ &= \sqrt{C'}\sqrt{1 - (1 + \tau^2/C' - 2\tau/\sqrt{C'})} \\ &= \sqrt{C'}\sqrt{1 - (1 - \tau/\sqrt{C'})^2}. \end{aligned} \tag{3.2.12}$$

Now consider a flat spatial geometry: taking $k = 0$, we have

$$\dot{a}^2 - C'/a^2 = 0.$$

Solving the differential equation yields

$$a(t) = \sqrt{2}\sqrt{\sqrt{C'}t} = (4C')^{1/4}t^{1/2}. \tag{3.2.13}$$

Then we consider the hyperboloid spatial geometry: taking $k = -1$, we have

$$\dot{a}^2 - C'/a^2 - 1 = 0. \tag{3.2.14}$$

Solving the differential equation yields

$$\begin{aligned} a(\tau) &= \sqrt{\tau^2 + 2\tau\sqrt{C'}} \\ &= \sqrt{C'}\sqrt{\tau^2/C' + 2\tau/\sqrt{C'}} \\ &= \sqrt{C'}\sqrt{(1 + \tau^2/C' + 2\tau/\sqrt{C'}) - 1} \\ &= \sqrt{C'}\sqrt{(1 + \tau/\sqrt{C'})^2 - 1}. \end{aligned} \tag{3.2.15}$$

For dust, first consider flat spatial geometry. Taking $k = 0$, we have the first order ordinary differential equation

$$\dot{a}^2 - C/a = 0. \tag{3.2.16}$$

Solving the differential equation yields the solution

$$a = (9C/4)^{1/3}\tau^{2/3}. \tag{3.2.17}$$

Then consider spherical spatial geometry. Taking $k = 1$, we have the first order ordinary differential equation

$$\dot{a}^2 - C/a + 1 = 0. \tag{3.2.18}$$

Solving the differential equation yields the solution

$$a = \frac{1}{2}C(1 - \cos\eta) \quad \text{and} \quad \tau = \frac{1}{2}C(\eta - \sin\eta), \tag{3.2.19}$$

where $\eta(\tau)$ is called the development angle. Next consider the hyperboloid spatial geometry. Taking $k = -1$, we have the first order ordinary differential equation

$$\dot{a}^2 - C/a - 1 = 0. \tag{3.2.20}$$

Solving the differential equation yields the solution

$$a = \frac{1}{2}C(\cosh\eta - 1) \quad \text{and} \quad \tau = \frac{1}{2}C(\sinh\eta - \eta), \tag{3.2.21}$$

where $\eta(\tau)$ is again called the development angle. The resulting six cases with expression for $a(\tau)$, i.e. equations 3.2.12, 14, 15,17,19, and 21 above, are referred to as the *Friedmann cosmology.*

3.3 Schwarzschild Solution

The Schwarzschild metric is a solution of Einstein's field equation which describes the exterior gravitational field of a static, spherically symmetric body.

Using spherical coordinates along with the time coordinate, the metric of an arbitrary static, spherically symmetric spacetime takes the form

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.3.1}$$

where f and h are functions of r. We shall see that the trivial orthonormal basis for the metric is

$$(e_0)_a = f^{1/2}(dt)_a \ , \tag{3.3.2}$$
$$(e_1)_a = h^{1/2}(dr)_a, \tag{3.3.3}$$
$$(e_2)_a = r(d\theta)_a, \tag{3.3.4}$$
$$(e_3)_a = r\sin\theta(d\phi)_a. \tag{3.3.5}$$

The ordinary derivative ∂_a defined on this coordinate system is then

$$\partial_{[a}(e_0)_{b]} = \frac{1}{2}f^{-1/2}f'(dr)_{[a}(dt)_{b]} \ , \tag{3.3.6}$$
$$\partial_{[a}(e_1)_{b]} = 0 \ , \tag{3.3.7}$$
$$\partial_{[a}(e_2)_{b]} = (dr)_{[a}(d\theta)_{b]} \ , \tag{3.3.8}$$
$$\partial_{[a}(e_3)_{b]} = \sin\theta(dr)_{[a}(d\phi)_{b]} + r\cos\theta(d\theta)_{[a}(d\phi)_{b]} \ , \tag{3.3.9}$$

where $f' = df/dr$. Computing the Riemann tensor, we obtain the following expressions:

$$R_{ab01} = -R_{ab10} = \frac{d}{dr}[(fh)^{-1/2}f'](dr)_{[a}(dt)_{b]} \ , \tag{3.3.10}$$
$$R_{ab02} = -R_{ab20} = f^{-1/2}h^{-1}f'(d\theta)_{[a}(dt)_{b]} \ , \tag{3.3.11}$$
$$R_{ab03} = -R_{ab30} = f^{-1/2}h^{-1}f'\sin\theta(d\phi)_{[a}(dt)_{b]} \ , \tag{3.3.12}$$
$$R_{ab12} = -R_{ab21} = h^{-3/2}h'(dr)_{[a}(d\theta)_{b]} \ , \tag{3.3.13}$$
$$R_{ab13} = -R_{ab31} = \sin\theta h^{-3/2}h'(d\theta)_{[a}(dt)_{b]} \ , \tag{3.3.14}$$
$$R_{ab23} = -R_{ab32} = 2(1-h^{-1})\sin\theta(d\theta)_{[a}(d\phi)_{b]} \ . \tag{3.3.15}$$

Using the above coefficients of the Riemann tensor, we can then compute the Ricci tensor. Equating the Ricci tensor to zero yields the vacuum Einstein equation for a static, spherically symmetric spacetime, defined as follows,

$$0 = R_{00} = R^1_{010} + R^2_{020} + R^3_{030} = \frac{1}{2}(fh)^{-1/2}\frac{d}{dr}[(fh)^{-1/2}f'] + (rfh)^{-1}f' \ , \tag{3.3.16}$$
$$0 = R_{11} = -\frac{1}{2}(fh)^{-1/2}\frac{d}{dr}[(fh)^{-1/2}f'] + (rh^2)^{-1}h' \ , \tag{3.3.17}$$
$$0 = R_{22} = R_{33} = -\frac{1}{2}(rfh)^{-1}f' + \frac{1}{2}(rh^2)^{-1}h' + r^{-2}(1-h^{-1}) \ , \tag{3.3.18}$$

where $R_{\mu\nu} = R_{ab}(e_\mu)^a(e_\nu)^b$. Note that all off-diagonal components of the Ricci curvature vanish to zero. Now, adding Equations 3.3.16 and 3.3.17 yields

$$\begin{aligned} \frac{f'}{rfh} + \frac{h'}{rh^2} &= 0 \\ \frac{f'}{f} + \frac{h'}{h} &= 0 \ , \end{aligned} \tag{3.3.19}$$

implying that

$$f = Kh^{-1} \ , \tag{3.3.20}$$

where K is a constant. We can re-scale the unit of the time coordinate to allow $K = 1$, i.e. $f = h^{-1}$ and thus $h'f^2 = -f'$. Thus Equation 3.3.18 becomes

$$\begin{aligned} 0 &= -\frac{1}{2}r^{-1}ff' + \frac{1}{2}r^{-1}f^2h' + r^{-2}(1-f) \\ &= -\frac{1}{2}r^{-1}f' - \frac{1}{2}r^{-1}f' + r^{-2}(1-f) \\ &= -r^{-1}f' + r^{-2}(1-h^{-1}) \\ 0 &= -f' + \frac{1-f}{r}. \end{aligned} \tag{3.3.21}$$

That is,

$$\frac{d}{dr}(rf) = f + rf' = f + 1 - f = 1. \tag{3.3.22}$$

Solving this separable differential equation yields

$$rf = r + C \text{ (for some constant } C\text{)}, \tag{3.3.23}$$

$$f = 1 + \frac{C}{r}. \tag{3.3.24}$$

Thus the Schwarzschild metric of the vacuum Einstein field equation for static, spherically symmetric spacetime is

$$ds^2 = -\left(1 + \frac{C}{r}\right)dt^2 + \left(1 + \frac{C}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.3.25}$$

$$= -\left(1 + \frac{C}{r}\right)dt^2 + \left(1 + \frac{C}{r}\right)dr^2 + r^2 d\Omega. \tag{3.3.26}$$

3.4 Application and Conclusion

Cosmology is the study of the origin, evolution, and eventual fate of the universe. It is an overview of all physical laws existing in this universe. As more complete solutions to Einstein's field equations are found, our understanding of cosmology becomes more thorough. In many way, the study of cosmology can directly affect our daily life. The universe may seem fairly empty, but it contains a great amount of roaming objects, traveling through space based on the laws of physics, each capable of affecting the stability of the Earth. On the other hand, the universe is also full of possibilities. Human society relies heavily on natural resources and forms of energy. An increased capability in cosmological studies may allow for better preventions of potential celestial disasters and discoveries of new resources.

As mentioned in previous sections, although classical physics provides a valid and accurate approximation for the physical motions and interactions of objects at the local scale, it has its fundamental flaws and deficiencies when applied to the cosmic scale or at the quantum scale. In cosmology, spectroscopy, or the study of the emission of electromagnetic radiation by matter, is among the most informative and consistent ways of measurements. Light is the fastest traveling particle in the universe, thus allowing us to obtain data about very distant galaxies. However, classical theories of physics fail to explain the motion of photons. In order to effectively analyze the data taken from cosmic spectroscopy, scientists must make use of the general theory of relativity. Exact solutions to Einstein's field equation, such as Friedman's solutions of the Robertson-Walker model and the Schwarzschild solutions, provide insights to the general spatial geometry of our universe, and thus allows us to make decent approximations for motions and interactions of matter at the cosmic level. Finding more exact solutions to Einstein's field equation is a clear path to fully understanding the cosmology of our universe.

References

[1] E. Abbena, S. Salamon and A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 3rd edition, Capman and Hall/CRC, 2006.

[2] S. M. Carroll, *Lecture Notes on General Relativity*, http://arxiv.org/abs/gr-qc/9712019.

[3] R. Millman and G. Parker, *Elements of Differential Geometry*, Prentice-Hall, Inc., 1977.

[4] J. Oprea, *Differential Geomtry and Its Applications*, Prentice-Hall, Inc., 2000.

[5] N. T. Redd, *Einstein's Theory of General Relativity*, space.com contributor, 2016.

[6] R. M. Wald, *General Relativity*, The University of Chicago Press, 1984.

[7] E. J. Chaisson, *Pratical Applications of Cosmology to Human Society*, The Natural Science, vol 6, pp. 767-596, 2014.