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Anticommutative Associative Algebras And The Binomial Theorem

Ashley Scurlock

Anticommutative Associative Algebras And The Binomial Theorem

Ashley M. Scurlock

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DATE APPROVED

__________ ______________________________

Jeb Collins, Ph.D.
thesis advisor

__________ ______________________________

Randall D. Helmstutler, Ph.D.
committee member

__________ ______________________________

J. Larry Lehman, Ph.D.
committee member

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Abstract

We examine the binomial theorem and its components in a noncommutative associative algebra. Specifically, we examine the relationship between the 1-binomial and -1-binomial coefficient, as well as exploring alternatives for the exponential identity for non-commutative and anticommutative elements. Through this investigation we found that the 1-binomial can be mapped to the -1-binomial and that the relationship could be used to prove a defined alternative for the exponential identity for anticommutative elements.

Background

1 Algebra

1.1 Noncommutative Algebra

The motivation behind this research is to investigate what happens when commutativity is removed from associative algebra since there are well known number systems that don't have commutativity. To do so we explored what happens to certain functions like the exponential function when commutativity is no longer there. Through this investigation, we were able to isolate at what point commutativity was needed for the identity to hold true. This led to us exploring the binomial theorem and its alternatives to see if there is a way to recreate the exponential identity without requiring commutativity.

Definition 1.1. An *algebra* A_0 is a vector space with a bilinear map $* : A_0 \times A_0 \to A_0$ referred to as multiplication and a variable $1 \in A_0$ such that for $a \in A_0$, $1 * a = a * 1 = a$. Usually we abbreviate $a * b$ as ab.

Multiplication is an integral part of algebra, and how multiplication is defined plays an essential part in whether or not variables are commutative.

Definition 1.2. Elements A and $B \in A_0$ are *q-commutative* if for a $q \in \mathbb{R}$, $AB = qBA$. There are two notable cases where $q = 1$ and $q = -1$. When $AB = BA$ then A and B are *commutative* and when $AB = -BA$ the elements A and B are *anticommutative*.

Commutativity is the founding assumption for many mathematical theorems. When it is removed it can be particularly restricting. However, if there was a way to link the non-commutative case with the commutative case, in theory, instances where commutativity does not exist would be much easier to solve.

1.2 Quaternions, Matrices, Rings

Within the sphere of noncommutative algebras there exists quaternions, matrices, and non-commutative rings. Quaternions are a special kind of algebra called normed division algebra.

Definition 1.3. A *normed division algebra* is an algebra A_0 that has a norm defined on it.

Quaternions are one of the four normed division algebras: real numbers, complex numbers, quaternions and octonions. Each normed division algebra is a number system that quantifies a different number of dimensions. Quaternions, as the name suggests, are a 4-dimensional number system that are used to describe the rotation of a 3-dimensional object.

Definition 1.4. *Quaternions* ($\mathbb{H}$) are comprised of a real component and three imaginary components, i, j, and k, with the relationships

$$i^2 = j^2 = k^2 = ijk = -1.$$
$$jk = i = -kj, \qquad ki = j = -ik, \qquad ij = k = -ji.$$

Quaternion multiplication is distinctly noncommutative. To define it let $A, B \in \mathbb{H}$ where $A = a + bi + cj + dk$, $B = w + xi + yj + zk$ and $a, b, c, d, w, x, y, z \in \mathbb{R}$. Then the product AB [4] is defined as,

$$AB = (aw - bx - cy - dz) + (ax + bw + cz - dy)i + (ay + cw + dx - bz)j + (az + dw + cx - yb)k.$$

It should be noted that there exists a subset of quaternions that anticommute. For every imaginary quaternion there exists a set of orthogonal quaternions that anticommute with it.

Definition 1.5. For $b, c, d, x, y, z \in \mathbb{R}$ two quaternions $A = bi+cj+dk$ and $B = xi+yj+zk$ are *orthogonal* if $bx + cy + dz = 0$.

However, it should be noted that for $q \neq -1$, $q \neq 1$, and $q \neq 0$ there are no quaternions that $q-$commute with each other.

Example 1.1 Consider orthogonal quaternions $A = 2i - 4j + 3k$ and $B = 3i + 3j + 2k$. Then

$$\begin{aligned} AB &= (2i - 4j + 3k)(3i + 3j + 2k) \\ &= -17i + 5j + 18k \\ &= -(17i - 5j - 18k) \\ &= -(3i + 3j + 2k)(2i - 4j + 3k) \\ &= -BA \end{aligned}$$

Another noncommutative algebra that contains anticommutative elements is matrices. A matrix is a rectangular array of values.

Example 1.2. Consider matrices $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then,

$$\begin{aligned} CD &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= -DC \end{aligned}$$

Anticommutativity will later be shown to be a unique case in part because when -1 is squared it becomes 1. An example of an identity that doesn't hold when commutativity is removed is $e^A e^B = e^{A+B}$, which will be further referred to as the exponential identity. To get a better understanding of this theorem, it is necessary to understand what the exponential of a quaternion or matrix looks like.

Definition 1.6. The series expansion for e^A where A is an element of any algebra is, [4]

$$e^A = \sum_{\ell=0}^{\infty} \frac{A^\ell}{\ell!}.$$

Where the series always converges, so the exponential of A is well-defined.

2 The Binomial Theorem and q-Binomial Theorem

2.1 Binomial Theorem

The binomial theorem is a quick and easy way to find the expanded powers of a binomial.

Theorem 2.1. Given a binomial to a power k where k is a nonnegative integer. The *binomial theorem* describes the binomial's polynomial expansion as,

$$(x+y)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} y^\ell = \sum_{\ell=0}^{k} \binom{k}{\ell} x^\ell y^{k-\ell}$$

The theorem itself has two distinct components. The first component, $\binom{k}{\ell}$, is referred to as the binomial coefficient. This element describes the coefficient of each of the terms in the expanded polynomial.

Definition 2.1. [1] For every $k, \ell \in N$ where $\ell \leq k$,

$$\binom{k}{\ell} = \frac{k!}{\ell!(k-\ell)!}.$$

Some of the many notable characteristics of the binomial coefficient is that it is center symmetric [2] and can be defined recursively.

Theorem 2.2. [1] For every $k, \ell \in N$ where $\ell \leq k$,

$$\binom{k}{\ell} = \binom{k-1}{\ell} + \binom{k-1}{\ell-1}.$$

The second component of the binomial theorem, $x^{k-\ell}y^{\ell}$, naturally describes the distribution of the terms x and y when the binomial is expanded. However, this result is dependent on the combination of terms which in itself is dependent on commutativity. This is best shown in the following example.

Example 2.2. If $xy = yx$ then,

$$\begin{aligned}(x+y)^3 &= (x^2+2xy+y^2)(x+y)\\ &= x^3+2xyx+y^2x+x^2y+2xy^2+y^3\\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

In Example 2.2 is is clear that results of both the binomial coefficient and the $x^{k-\ell}y^{\ell}$ component rely on the fact that x and y are commutative.

2.2 Application for the Binomial Theorem

The binomial theorem is the building block of many other identities, one of which is the identity $e^A e^B = e^{A+B}$ where A and B are any two commutative elements in an algebra.

Definition 2.2. Let $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$ be two infinite sums where a_i and b_j are elements of an algebra. The *Cauchy product* of the two sums is defined as,

$$\sum_{i=0}^{\infty} a_i \sum_{j=0}^{\infty} b_j = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} a_\ell b_{k-\ell}.$$

Unlike the binomial theorem, the Cauchy product does not require commutativity for the identity to be true. However, if you look at the proof for the exponential identity it becomes clear that values A and B must commute.

Theorem 2.3. For a commutative A and B, $e^A e^B = e^{A+B}$.

Proof. If A and B are elements of a commutative algebra then $e^A = \sum_{\ell=1}^{\infty} \frac{A^\ell}{\ell!}$ and $e^B = \sum_{\ell=1}^{\infty} \frac{B^\ell}{\ell!}$. When A and B commute then,

$$\begin{aligned}
e^A e^B &= \sum_{\ell=0}^{\infty} \frac{A^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{B^\ell}{\ell!} && \text{(Definition 1.6)} \\
&= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{A^\ell}{\ell!} \frac{B^{k-\ell}}{(k-\ell)!} && \text{(Cauchy Product, Definition 2.2)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{k} \frac{k!}{\ell!(k-\ell)!} A^\ell B^{k-\ell} \\
&= \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} && \text{(Binomial Theorem, Definition 2.1)} \\
&= e^{A+B}.
\end{aligned}$$

Therefore, when A and B commute then $e^A e^B = e^{A+B}$. □

Since the binomial theorem requires commutativity, when it is removed the exponential identity no longer holds [1]. Before trying to develop an alternative for the exponential identity, consider the existing variations of the binomial theorem.

2.3 Gaussian Binomial Coefficient

While the binomial theorem required commutativity there exists an alternative called the q-binomial theorem which requires q-commutativity.

Definition 2.3. For a q-commutative x and y the expanded binomial is defined as,

$$(x+y)^k = \sum_{\ell=0}^{k} \binom{k}{\ell}_q x^{k-\ell} y^k = \sum_{\ell=0}^{k} \binom{k}{\ell}_q x^\ell y^{k-\ell}. \tag{1}$$

Clearly the theorem is nearly identical to the binomial theorem, however, in this case the theorem uses what is referred to as the Gaussian coefficient.

Definition 2.4. For every $k, \ell \in N$ where $\ell \leq k$ the *Gaussian coefficient* is defined as

$$\binom{k}{\ell}_q = \lim_{p \to q} \left[\frac{[p^k - 1][p^{k-1} - 1] \cdots [p^{k-\ell+1} - 1]}{[p^\ell - 1][p^{\ell-1} - 1] \cdots [p - 1]} \right]. \tag{2}$$

Similar to the binomial coefficient the Gaussian coefficients are also center symmetric [2] and defined recursively. However, in this case it requires an extra variable $q^{k-\ell}$.

Theorem 2.4. The recursive equation for the Gaussian coefficient is defined as,

$$\binom{k}{\ell}_q = \binom{k-1}{\ell}_q + q^{k-\ell} \binom{k-1}{\ell-1}_q$$

3 Binomial and Gaussian Coefficients

As shown in Definition 1.3, commutativity is just a special case of q-commutativity where $q = 1$. If we solve for the Gaussian coefficient when $q = 1$ we expect to get the same result as in Theorem 2.2.

Example 3.1 When $q = 1$ clearly as shown below,

$$\binom{k}{\ell}_1 = \lim_{q\to 1}\left[\frac{[(q)^k-1][(q)^{k-1}-1]\cdots[(q)^{k-\ell+1}-1]}{[(q)^\ell-1][(q)^{\ell-1}-1]\cdots[(q)-1]}\right].$$

In this case the limit would be indeterminate. Thus we implement L'Hopital's rule to get the following:

$$\begin{aligned}\binom{k}{\ell}_1 &= \lim_{q\to 1}\left[\frac{[(q)^k-1][(q)^{k-1}-1]\cdots[(q)^{k-\ell+1}-1]}{[(q)^\ell-1][(q)^{\ell-1}-1]\cdots[(q)-1]}\right]\\ &= \lim_{q\to 1}\left[\frac{\frac{d}{dq}\left[[(q)^k-1][(q)^{k-1}-1]\cdots[(q)^{k-\ell+1}-1]\right]}{\frac{d}{dq}\left[[(q)^\ell-1][(q)^{\ell-1}-1]\cdots[(q)-1]\right]}\right]\\ &= \lim_{q\to 1}\left[\frac{\frac{d}{dq}[(q)^k-1]}{\frac{d}{dq}[(q)^\ell-1]}\right]\lim_{q\to 1}\left[\frac{\frac{d}{dq}[(q)^{k-1}-1]}{\frac{d}{dq}[(q)^{\ell-1}-1]}\right]\cdots\lim_{q\to 1}\left[\frac{\frac{d}{dq}[(q)^{k-\ell+1}-1]}{\frac{d}{dq}[(q)-1]}\right]\\ &= \lim_{q\to 1}\left[\frac{kq^{k-1}}{\ell q^{\ell-1}}\right]\lim_{q\to 1}\left[\frac{(k-1)q^{k-2}}{(\ell-1)q^{\ell-2}}\right]\cdots\lim_{q\to 1}\left[\frac{(k-\ell+1)q^{k-\ell}}{1}\right]\\ &= \left[\frac{[k][k-1]\cdots[k-\ell+1]}{[\ell][\ell-1]\cdots[1]}\right]\\ &= \frac{k!}{\ell!(k-\ell)!}\end{aligned}$$

The only other case in which we would get an indeterminate form is when $q = -1$ so we can implement a similar process. However, in this instance the elements $(q^x - 1)$ are only 0 when x is even.

In this case we cannot just apply L'Hopital's rule like we did in the previous example; we have to consider whether the exponents are even or odd since this determines whether or not the element will equal 0.

Definition 3.1. For a nonnegative integer a let a_e indicate the even value $a - 2 < a_e \leq a$ and a_o indicate the odd value $(a-2) < a_o \leq a$.

Lemma 3.1. For nonnegative integers a and b if a is odd or b is even then,

$$\binom{a}{b}_{-1} = \lim_{q\to -1}\left[\frac{(q^{a_e}-1)(q^{a_e-2}-1)\cdots(q^{a_e-b_e+2}-1)}{(q^{b_e}-1)(q^{b_e-2}-1)\cdots(q^2-1)}\right].$$

Proof. Let a and b be nonnegative integers where a is odd or b is even. Then based on the formula,

$$\binom{a}{b}_q = \lim_{p\to q}\left[\frac{[(q)^a-1][(q)^{a-1}-1]\cdots[(q)^{a-b+1}-1]}{[(q)^b-1][(q)^{b-1}-1]\cdots[(q)-1]}\right].$$

It is clear that the numerator has $[a-(a-b)] = b$ factors and the denominator has b factors. When b is even the numerator and denominator will have $\frac{b}{2}$ elements where the exponent is even. We know b is only odd when a is odd then the sequences $\{a-b+1, a-b, \cdots, a\}$ and $\{b, b-1, \ldots, 1\}$ will have the same number of even and odd values. Therefore,

$$\begin{aligned}
\binom{a}{b}_{-1} &= \lim_{q\to -1}\left[\frac{(q^a-1)(q^{a-1}-1)\cdots(q^{a-b+1}-1)}{(q^b-1)(q^{b-1}-1)\cdots(q-1)}\right] \\
&= \lim_{q\to -1}\left[\frac{(q^{a_e}-1)(q^{a_e-2}-1)\cdots(q^{a_e-b_e+2}-1)}{(q^{b_e}-1)(q^{b_e-2}-1)\cdots(q^2-1)}\right]\cdot \\
&\quad \lim_{q\to -1}\left[\frac{(q^{a_o}-1)(q^{a_o-2}-1)\cdots(q^{a_o-b_o+1}-1)}{(q^{b_o}-1)(q^{b_o}-1)\cdots(q-1)}\right] \text{(Separate into even and odd exponents)} \\
&= \lim_{q\to -1}\left[\frac{(q^{a_e}-1)(q^{a_e-2}-1)\cdots(q^{a_e-b_e+2}-1)}{(q^{b_e}-1)(q^{b_e-2}-1)\cdots(q^2-1)}\right]\left[\frac{(-2)^b}{(-2)^b}\right] \text{(Solve for the limit of the odd exponents)} \\
&= \lim_{q\to -1}\left[\frac{(q^{a_e}-1)(q^{a_e-2}-1)\cdots(q^{a_e-b_e+2}-1)}{(q^{b_e}-1)(q^{b_e-2}-1)\cdots(q^2-1)}\right].
\end{aligned}$$

Therefore when a is odd or b is even then $\binom{a}{b}_{-1} = \lim_{q\to -1}\left[\frac{(q^{a_e}-1)(q^{a_e-2}-1)\cdots(q^{a_e-b_e+2}-1)}{(q^{b_e}-1)(q^{b_e-2}-1)\cdots(q^2-1)}\right]$. □

Lemma 3.2. Let $a, b \in \mathbb{N}$ where a is even and b is odd. Then,

$$\binom{a}{b}_{-1} = \lim_{q\to -1}\left[\frac{(q^a-1)}{(q^{b-1}-1)}\right]\lim_{q\to -1}\left[\frac{(q^{a-1}-1)}{(q^b-1)}\right]\lim_{q\to -1}\left[\frac{(q^{a-2}-1)}{(q^{b-3}-1)}\right]\cdots\lim_{q\to -1}\left[\frac{(q^{a-b_O+1}-1)}{(q-1)}\right]$$

Then $\frac{b+1}{2}$ of the elements in the numerator have even exponents and $\frac{b-1}{2}$ elements in the denominator have even exponents.

Utilizing Lemmas 3.1 and 3.2 we can now condense the equation for $\binom{k}{\ell}_{-1}$. However, since the result is dependent on whether k and ℓ are odd or even we must consider two separate cases. For the ease of the reader consider the following notation:

Definition 3.2. Let the notation $a!!_E$ indicate the product of every even natural number less than or equal to a.

The notation in definition 3.2 is a variation of the existing double factorial which takes the product of every other value of an integer a represented as $a!!$.

Theorem 3.1. When k is odd or ℓ is even then,

$$\binom{k}{\ell}_{-1} = \frac{k!!_E}{\ell!!_E(k-\ell)!!_E}$$

Proof. Let $k, \ell \in \mathbb{N}$ where k is odd or ℓ is even. Then,

$$\begin{aligned}
\lim_{q\to -1}\binom{k}{\ell}_{-1} &= \lim_{q\to -1}\left[\frac{(q^k-1)(q^{k-1}-1)\cdots(q^{k-\ell+1}-1)}{(q^\ell-1)(q^{\ell-1}-1)\cdots(q-1)}\right] \\
&= \lim_{q\to -1}\left[\frac{(q^{k_e}-1)(q^{k_e-2}-1)\cdots(q^{k_e-\ell_e+2}-1)}{(q^{\ell_e}-1)(q^{\ell_e-2}-1)\cdots(q^2-1)}\right] \quad \text{(By Lemma 3.1)} \\
&= \lim_{q\to -1}\left[\frac{((k_e)q^{k_e-1})((k_e-2)q^{k_e-3})\cdots((k_e-\ell_e+2)q^{k_e-\ell_e+1})}{((\ell_e)q^{\ell_e-1})((\ell_e-2)q^{\ell_e-3})\cdots(2q)}\right] \\
&= \left[\frac{(k_e)(k_e-2)\cdots(k_e-\ell_e+2)}{(\ell_e)(\ell_e-2)\cdots(2)}\right] \quad \text{(Evaluate the limit)} \\
&= \frac{(k_e)!!}{(\ell_e)!!(k_e-\ell_e)!!} \\
&= \frac{k!!_E}{\ell!!_E(k-\ell)!!_E} \qquad \text{(By definition 3.1 defining the syntax)}
\end{aligned}$$

Therefore when k is even or ℓ is odd, $\binom{k}{\ell}_{-1} = \frac{k!!_E}{\ell!!_E(k-\ell)!!_E}$. □

Now by definition 3.1 we get the following lemmas.

Lemma 3.3. For k, $\ell \in \mathbb{N}$ where k and ℓ are even,

$$\binom{k}{\ell}_{-1} = \binom{k+1}{\ell+1}_{-1}$$

Proof. Consider for even values k and ℓ the -1-binomial coefficient $\binom{k}{\ell}_{-1}$. Then,

$$\begin{aligned}
\binom{k}{\ell_{-1}} &= \frac{(k)!!_E}{((\ell)!!_E)(k-\ell)!!_E} \\
&= \frac{(k+1)!!_E}{((\ell+1)!!_E)((k+1)-(\ell+1))!!_E} \quad \text{(Property of } !!_E) \\
&= \binom{k+1}{\ell+1}_{-1}
\end{aligned}$$

Therefore, $\binom{k}{\ell}_{-1} = \binom{k+1}{\ell+1}_{-1}$. □

Lemma 3.4. For k, $\ell \in \mathbb{N}$ where k and ℓ are even,

$$\binom{k}{\ell}_{-1} = \binom{k+1}{\ell}_{-1}$$

Proof. Consider the (-1)-binomial coefficient $\binom{k}{\ell}_{-1}$. Then,

$$\begin{aligned}
\binom{k}{\ell}_{-1} &= \frac{(k)!!_E}{((\ell)!!_E)(k-\ell))!!_E} \\
&= \frac{(k+1)!!_E}{((\ell)!!_E)((k+1)-\ell))!!_E} \quad \text{(Property of } !!_E) \\
&= \binom{k+1}{\ell}_{-1}
\end{aligned}$$

Thus, by extension we can assume that the following is true when k is even and ℓ is odd or when k is odd and ℓ is even:

$$\binom{k+\ell-1}{2(\ell)}_{-1} = \binom{(k+1)+\ell-1}{2(\ell)}_{-1}$$

. □

However, there is one special case for an even k and an odd ℓ.

Theorem 3.2. When k is even and ℓ is odd then,

$$\binom{k}{\ell}_{-1} = 0$$

Proof. Let $k, \ell \in \mathbb{N}$ where $\ell < k$. To prove $\binom{k}{\ell}_{-1} = 0$ we must consider the case where $\ell = 1$ or $\ell = k-1$ and the case where $1 < \ell_O < k-1$.

Case 1: $\ell = 1$ or $\ell = k-1$. Let $\ell = 1$ then,

$$\begin{aligned}\binom{k}{1}_{-1} &= \lim_{q\to -1}\left[\frac{q^k-1}{q-1}\right]\\ &= \frac{1-1}{(-1)-1}\\ &= 0\end{aligned}$$

Since the q-binomial coefficients are center-symmetric then $\lim_{q\to -1}\binom{k}{1}_{-1} = \lim_{q\to -1}\binom{k}{k-1}_{-1} = 0$.

Case 2: Let $\ell < k-1$ and $\ell > 1$,

$$\begin{aligned}\binom{k}{\ell}_{-1} &= \lim_{q\to -1}\left[\frac{(q^k-1)(q^{k-1}-1)\cdots(q^{k-\ell+1}-1)}{(q^\ell-1)(q^{\ell-1}-1)\cdots(q-1)}\right]\\ &= \lim_{q\to -1}\left[\frac{(q^k-1)}{(q^{\ell-1}-1)}\right]\lim_{q\to -1}\left[\frac{(q^{k-1}-1)}{(q^\ell-1)}\right]\lim_{q\to -1}\left[\frac{(q^{k-2}-1)}{(q^{\ell-3}-1)}\right]\cdots\\ &\qquad \cdots\lim_{q\to -1}\left[\frac{(q^{k-\ell+1}-1)}{(q-1)}\right] \quad \text{(Lemma 3.2)}\\ &= \lim_{q\to -1}\left[\frac{(q^k-1)}{(q^{\ell-1}-1)}\right]\left[\frac{-2}{-2}\right]\lim_{q\to -1}\left[\frac{(q^{k-2}-1)}{(q^{\ell-3}-1)}\right]\cdots\left[\frac{0}{-2}\right] \quad \text{(Find the limit of the variables with odd exponents)}\\ &= \lim_{q\to -1}\left[\frac{(q^k-1)}{(q^{\ell-1}-1)}\right]\lim_{q\to -1}\left[\frac{(q^{k-2}-1)}{(q^{\ell-3}-1)}\right]\cdots 0\\ &= 0\end{aligned}$$

Therefore, $\binom{k}{\ell}_{-1} = 0$. □

Now that we have rewritten the formula for the -1-binomial coefficient we can better show how it connects to the 1-binomial coefficient.

3.1 Pascal's Triangle

The binomial coefficients can be configured into a triangular array known as Pascal's triangle. Similarly, the $q-$binomial coefficients can be configured into a triangular array known as the $q-$Pascal triangle. We can better visualize the relationship between the -1-binomial coefficient and the 1-binomial coefficient by comparing the Pascal and $q-$Pascal triangles.

Example 3.2. The following triangular array is the $-1-$Pascal's triangle. Notice that on every other diagonal the values repeat. This is shown in Lemmas 3.3 and 3.4.

k																	
0									1								
1								1		1							
2							1		0		1						
3						1		1		1		1					
4					1		0		2		0		1				
5				1		1		2		2		1		1			
6			1		0		3		0		3		0		1		
7		1		1		3		3		3		3		1		1	
8	1		0		4		0		6		0		4		0		1

ℓ

Now we will compare this triangle to Pascal's triangle.

Example 3.3.
Pascal's triangle

k									
0					1				
1				1		1			
2			1		2		1		
3		1		3		3		1	
4	1		4		6		4		1

ℓ

-1-Pascal's triangle

k																	
0									1								
1								1		1							
2							1		0		1						
3						1		1		1		1					
4					1		0		2		0		1				
5				1		1		2		2		1		1			
6			1		0		3		0		3		0		1		
7		1		1		3		3		3		3		1		1	
8	1		0		4		0		6		0		4		0		1

ℓ

Notice that the highlighted values correspond to the highlighted values of the same color. We will prove this relationship using the following lemma.

Lemma 3.1.1. If $\ell, k \in \mathbb{N}$ then $\binom{k}{\ell} = \binom{2k}{2\ell}_{-1}$.

Proof. Let ℓ and k be nonnegative integers.. First, to show the relationship between the factorial and double factorial consider,

$$\begin{aligned}(2\ell)!! &= (2\ell)(2\ell-2)(2\ell-4)\cdots(2)\\ &= [2(\ell)][2(\ell-1)][(2(\ell-2)]\cdots(2)\\ &= 2^{\ell}(\ell)!\end{aligned}$$

Furthermore, using that relationship we can show that,

$$\begin{aligned}\binom{k}{\ell} &= \frac{k!}{\ell!(k-\ell)!}\\ &= \left[\frac{k!}{\ell!(k-\ell)!}\right]\left[\frac{2^k}{2^{\ell}2^{\ell-k}}\right]\\ &= \frac{2^k(k)!}{2^{\ell}(\ell)!2^{k-\ell}(k-\ell)!}\\ &= \binom{2k}{2\ell}_{-1} \qquad \text{(By Theorem 3.1).}\end{aligned}$$

Therefore for all nonnegative integers ℓ and k $\binom{k}{\ell} = \binom{2k}{2\ell}_{-1}$. □

These relationships serve as the foundation for an exponential identity for anticommutative values.

4 Variations of The Exponential Identity

4.1 Background

In our research, we found two alternatives to the exponential identity that hold in a noncommutative algebra: the Campbell-Baker-Hausdorff formula and a Dr. Walter Wyss's Corollary 3 [5].

Theorem 4.1. [3] For a noncommutative A and B the Campbell-Baker-Hausdorff formula is the solution for Z in the equation $e^A e^B = e^Z$ where
$Z = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]]\ldots$

The Campbell-Baker-Hausdorff formula is particularly helpful in Lie algebra, where it is common to transform variables using the natural log which makes the theorem much easier to work with.

However, there are some situations where the exponential identity is used in a situation where it wouldn't be beneficial to take the natural log. Because of this, Wyss discovered an alternative that utilizes the identity as we know it plus an extra element. In that extra element he uses the following notation:

Definition 4.1. [5] For two elements A and q in any algebra the function dq is defined such that $dq(A) = qA - Aq$.

Definition 4.2. [5] The variable D_k is defined by the recursive statement $D_{k+1} = dq(A^k) + (A + dB)D_k$ with $D_0 = D_1 = 0$.

These elements help make up the essential non-commutative part of his equation.

Corollary 4.1. [5] Let A and B be elements in a non-commutative, associative algebra with identity. Then,

$$e^{A+B} = e^A e^B + \sum_{k=0}^{\infty} \frac{1}{k!} D_k e^B$$

Definition 4.3. For any $a \in \mathbb{N}$ let a_E indicate that a is even. Likewise, let a_O indicate that a is odd.

Defining this notation in Definition 4.3 is significant as it makes it easy for the ready to identify which variables are even and which are odd. This is important because the lemmas in section 4.2 and theorem in section 4.3 heavily depend on whether the variables are even or odd.

4.2 Lemmas

For the sake of the reader, in the Lemmas 4.2, 4.2.1, 4.2.2, and 4.2.3 let p and q be two anticommutative elements in an associative algebra and the following values M, M_2, J, N, N_2, and L be defined as:

$$M = \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O-\ell_O)+1} \qquad N = \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E+1} q^{(\ell_E)+1} p^{k_O-\ell_E}$$

$$M_2 = \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O-\ell_O)+1} \qquad N_2 = \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-2}{2(\ell_O)-2}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}$$

$$J = \binom{k_O+1}{2}_{-1} 2qp^{k_O} \qquad L = \binom{(k_O+1)+(k_O+1)-1}{(k_O+1)+(k_O+1)-1}_{-1} 2^{k_O} q^{k_O} p^{(k_O+1)-k_O}$$

Lemma 4.2.1. For the values N, N_2, L defined above,

$$N = N_2 + L.$$

Proof. To prove that $N = N_2 + L$, note that $\ell_E = \ell_O - 1$. Then,

$$N = \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O + \ell_E - 1}{2(\ell_E)}_{-1} 2^{\ell_E+1} q^{\ell_E+1} p^{k_O-\ell_E} = \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O} \binom{k_O + (\ell_O - 1) - 1}{2(\ell_O - 1)}_{-1} 2^{\ell_O} q^{\ell_O} p^{k_O-(\ell_O-1)}$$

$$= \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O} \binom{k_O + \ell_O - 2}{2(\ell_O) - 2}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}$$

$$= \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O + \ell_O - 2}{2(\ell_O) - 2}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O} + \binom{k_O + k_O - 2}{2(k_O) - 2}_{-1} 2^{k_O} q^{k_O} p^{(k_O+1)-k_O}$$

$$= N_2 + \binom{k_O + k_O - 2}{2(k_O) - 2}_{-1} 2^{k_O} q^{k_O} p^{(k_O+1)-k_O}.$$

Note that $\binom{k_O+k_O-2}{2(k_O)-2}_{-1} = 1$ because the upper and lower elements are equal. Therefore, $\binom{k_O+k_O-2}{2(k_O)-2}_{-1} 2^{k_O} q^{k_O} p^{(k_O+1)-k_O} = L$ and thus $N = N_2 + L$. □

Lemma 4.2.2. For the values M, M_2, J defined above,

$$2qp^{k_O} + M = M_2 + J.$$

Proof. First to prove this lemma we must remove the initial value in the summation M.

$$M = \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O + \ell_O - 1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O-\ell_O)+1}$$

$$= \binom{k_O + 1 - 1}{2(1)}_{-1} 2qp^{k_O} + \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O + \ell_O - 1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O-\ell_O)+1}$$

$$= \binom{k_O + 1 - 1}{2(1)}_{-1} 2qp^{k_O} + M_2.$$

Then consider the sum of $2qp^{k_O}$ and the first value of the summation M,

$$2qp^{k_O} + \binom{k_O + 1 - 1}{2(1)}_{-1} 2qp^{k_O} = \binom{k_O}{1}_{-1} 2qp^{k_O} + \binom{k_O}{2}_{-1} 2qp^{k_O} \left(\binom{k_O}{1}_{-1} = 1 \text{ by Lemmas 3.1.1 and 3.3} \right)$$

$$= \left[\binom{k_O}{2}_{-1} + (-1)^{k_O-1} \binom{k_O}{1}_{-1} \right] 2qp^{k_O}$$

$$= \binom{k_O + 1}{2}_{-1} 2qp^{k_O} \qquad \text{(by Theorem 2.7.)}$$

$$= J.$$

Therefore, $2qp^{k_O} + M = M_2 + J$. □

Lemma 4.2.3. For values M_2 and N_2 defined above,

$$M_2 + N_2 = \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}.$$

Proof. Consider the summations M_2 and N_2. Then,

$$\begin{aligned}
M_2 + N_2 &= \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O} + \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-2}{2(\ell_O)-2}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}. \\
&= \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \left[\binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} + \binom{k_O+\ell_O-2}{2(\ell_O)-2}_{-1}\right] 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O} \\
&= \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \left[\binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} + (-1)^{(k_O-\ell_O)}\binom{k_O+\ell_O-1}{2(\ell_O)-1}_{-1}\right] 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}. \text{(By Lemma 3.3)} \\
&= \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} (-1)^{(k_O-\ell_O)} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}. \quad \text{(By Theorem 2.7.)}
\end{aligned}$$

Therefore, $M_2 + N_2 = \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}$. □

Lemma 4.2. Let k_O and ℓ_O be odd values in $\mathbb{N}$. Then,

$$2qp^{k_O} + M + N = \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}.$$

Proof. To prove Lemma 4.2 use the definitions of M and N as defined above,

$$\begin{aligned}
2qp^{k_O} + M + N &= 2qp^{k_O} + M + (N_2 + L) \qquad \text{(Lemma 4.2.1)} \\
&= (M_2 + J) + N_2 + L \qquad \text{(Lemma 4.2.2)} \\
&= (M_2 + N_2) + J + L \\
&= \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O} + J + L \quad \text{(Lemma 4.2.3)} \\
&= \sum_{\substack{\ell_O=3 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O} \\
&\quad + \binom{(k_O+1)+1-1}{2}_{-1} 2qp^{k_O} + \binom{(k_O+1)+(k_O+1)-1}{(k_O+1)+(k_O+1)-1}_{-1} 2^{k_O} q^{k_O} p \\
&= \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}.
\end{aligned}$$

Therefore, $2qp^{k_O} + N + M = \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O}$.

□

For the sake of the reader, in the Lemma 4.3 let the following values Y, Y_2, X, and Z be defined as:

$$Y = \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} 2^{(\ell_O)+1} q^{(\ell_O)+1} p^{k_E-\ell_O} \qquad X = \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E-\ell_E)+1}$$

$$Y_2 = \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)-1}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} \qquad Z = \binom{2(k_E)-1}{2(k_E)-1}_{-1} 2^{k_E} q^{k_E} p^{(k_E-k_E)+1}$$

Lemma 4.3. For values X and Y defined above,

$$X + Y = \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E} \binom{(k_E+1)+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E}.$$

Proof. To prove Lemma 4.3, first we must show that $Y = Y_2 + Z$.

$$
\begin{aligned}
Y &= \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} 2^{(\ell_O)+1} q^{(\ell_O)+1} p^{k_E-\ell_O} \\
&= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E} \binom{k_E+(\ell_E-1)-1}{2(\ell_E-1)}_{-1} 2^{(\ell_E-1)+1} q^{(\ell_E-1)+1} p^{k_E-(\ell_E-1)} \qquad (\text{Let } \ell_O = \ell_E - 1) \\
&= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E} \binom{k_E+\ell_E-2}{2(\ell_E)-2}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} \\
&= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E} \binom{k_E+\ell_E-1}{2(\ell_E)-1}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} \qquad (\text{By Lemma 3.3}) \\
&= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)-1}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} + \binom{2(k_E)-1}{2(k_E)-1}_{-1} 2^{k_E} q^{k_E} p^{(k_E-k_E)+1} \\
&= Y_2 + Z
\end{aligned}
$$

Next we evaluate $X + Y_2$,

$$
\begin{aligned}
X + Y_2 &= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E-\ell_E)+1} + \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)-1}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} \\
&= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \left[\binom{k_E+\ell_E-1}{2(\ell_E)}_{-1} + (-1)^{(k_e-\ell_E)} \binom{k_E+\ell_E-1}{2(\ell_E)-1}_{-1} \right] 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} \\
&= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{(k_E+1)+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E}. \qquad (\text{By Theorem 2.7.})
\end{aligned}
$$

Finally, we can find $(X + Y_2) + Z$,

$$
\begin{aligned}
(X + Y_2) + Z &= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{(k_E+1)+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} + \binom{2(k_E)-1}{2(k_E)-1}_{-1} 2^{k_E} q^{k_E} p^{(k_E+1)-k_E} \\
&= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E} \binom{(k_E+1)+\ell_E-1}{2(\ell_E)}_{-1} 2^{k_E} q^{\ell_E} p^{(k_E+1)-\ell_E}
\end{aligned}
$$

Therefore, $X + Y = \displaystyle\sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E} \binom{(k_E+1)+\ell_E-1}{2(\ell_E)}_{-1} 2^{k_E} q^{\ell_E} p^{(k_E+1)-\ell_E}.$ □

4.3 Anticommutative Exponential Identity

Theorem 4.2. Let p and q be two elements of an associative algebra that anticommute. Then,

$$D_k = \sum_{\ell=1}^{k-1} \binom{k+\ell-1}{2\ell}_{-1} (-1)^{k\ell} 2^\ell q^\ell p^{k-\ell}.$$

Proof. We will proceed by induction. Let p and q be two elements of an associative algebra such that $pq = -qp$.

Base Case: Given that $D_0 = D_1 = 0$ by Definition 4.2 let $k = 2$. Then,

$$D_2 = dq(p^1) + (p + dq)D_1 = (qp - pq) + 0 = 2qp.$$

Moreover,

$$\sum_{\ell=1}^{2-1} \binom{2+1-1}{2(1)}_{-1} (-1)^{2(1)} 2^1 q^1 p^{2-1} = \binom{2}{2}_{-1} (-1)^2 2qp = (1)(1)2qp = 2qp.$$

Hence the theorem is true for $k = 2$.

Induction step: Assume that

$$D_k = \sum_{\ell=1}^{k-1} \binom{k+\ell-1}{2\ell}_{-1} (-1)^{k\ell} 2^\ell q^\ell p^{k-\ell}$$

is true. To prove that the $k+1$ case is true we must consider the following two cases. The first case is when k is even and the second is when k is odd.

Case 1: k is even. To denote that k is even let $k = k_E$. Then assume that $D_{k_E} = \sum_{\ell=1}^{k_E-1} \binom{k_E+\ell-1}{2\ell}_{-1} (-1)^{k_E\ell} 2^\ell q^\ell p^{k_E-\ell}$. We then see that,

$$\begin{aligned}
(p + dq)D_{k_E} &= (p + dq) \sum_{\ell=1}^{k_E-1} \binom{k_E+\ell-1}{2\ell}_{-1} (-1)^{k_E\ell} 2^\ell q^\ell p^{k_E-\ell} \\
&= p \left[\sum_{\ell=1}^{k_E-1} \binom{k_E+\ell-1}{2\ell}_{-1} (-1)^{k_E\ell} 2^\ell q^\ell p^{k_E-\ell} \right] \\
&\quad + q \left[\sum_{\ell=1}^{k_E-1} \binom{k_E+\ell-1}{2\ell}_{-1} (-1)^{k_E\ell} 2^\ell q^\ell p^{k_E-\ell} \right] \\
&\quad - \left[\sum_{\ell=1}^{k_E-1} \binom{k_E+\ell-1}{2\ell}_{-1} (-1)^{k_E\ell} 2^\ell q^\ell p^{k_E-\ell} \right] q.
\end{aligned}$$

By anticommutative law we can move the p and q to combine like terms. However, whether or not the sign changes is dependent on whether or not the exponents of p and q

are odd or even. When the exponent of q^{ℓ_O} is odd then the sign does change. For example, $pq^{\ell_O}p = (-1)^{\ell_O}q^{\ell_O}p^2 = -q^{\ell_O}p^2$. Likewise, when the exponent ℓ_E is even the sign stays the same. For example, $pq^{\ell_E}p = (-1)^{\ell_E}q^{\ell_E}p^2 = q^{\ell_O}p^2$. To further simplify we must split the summations into the odd and even ℓ values. Then,

$$
\begin{aligned}
(p+dq)D_{k_E} = & \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)}_{-1} (-1)^{(k_E\ell_E)} 2^{\ell_E} q^{\ell_E} p^{(k_E-\ell_E)+1} \\
& + \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} (-1)(-1)^{(k_E\ell_O)} 2^{\ell_O} q^{\ell_O} p^{(k_E-\ell_O)+1} \\
& + \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)}_{-1} (-1)^{(k_E\ell_E)} 2^{\ell_E} q^{(\ell_E)+1} p^{k_E-\ell_E} \\
& + \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} (-1)^{(k_E\ell_O)} 2^{\ell_O} q^{(\ell_O)+1} p^{k_E-\ell_O} \\
& - \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)}_{-1} (-1)^{(k_E\ell_E)} 2^{\ell_E} q^{(\ell_E)+1} p^{k_E-\ell_E} \\
& - \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} (-1)(-1)^{(k_E\ell_O)} 2^{\ell_O} q^{(\ell_O)+1} p^{k_E-\ell_O}.
\end{aligned}
$$

Clearly the last two even summations cancel and the expressions can be simplified such that,

$$
\begin{aligned}
(p+dq)D_{k_E} = & \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E-\ell_E)+1} \\
& - \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_E-\ell_O)+1} \\
& + \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{(\ell_O)+1} p^{k_E-\ell_O} \\
& + \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{(\ell_O)+1} p^{k_E-\ell_O}.
\end{aligned}
$$

Now we can see that the last two odd summations can be combined such that,

$$\begin{aligned}
(p+dq)D_{k_E} = & \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E-2} \binom{k_E+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} \\
& - \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_E+1)-\ell_O} \\
& + \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} 2^{(\ell_O)+1} q^{(\ell_O)+1} p^{k_E-\ell_O} \\
= & - \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{k_E+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_E+1)-\ell_O} \qquad \text{(By Lemma 4.3)} \\
& + \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E} \binom{(k_E+1)+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E} \\
= & - \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_E-1} \binom{(k_E+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_E+1)-\ell_O} \qquad \text{(By Lemma 3.4)} \\
& + \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_E} \binom{(k_E+1)+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_E+1)-\ell_E}.
\end{aligned}$$

To get D_{k+1} we add the $dq(p^{k_E})$ term which is equal to $qp^{k_E} - p^{k_E}q = qp^{k_E} - (-1)^{k_E} qp^{k_E} = 0$. Hence,

$$D_{k+1} = dq(p^{k_E}) + (p+dq)D_{k_E} = \sum_{\ell=1}^{(k+1)-1} \binom{(k_E+1)+\ell-1}{2(\ell)}_{-1} (-1)^{(k_E+1)\ell} 2^\ell q^\ell p^{(k_E+1)-\ell}.$$

Therefore, when the theorem is true for an even k then the theorem is true for $k+1$.

Case 2: k is odd Let $k = k_O$ Assume that $D_{k_O} = \sum_{\ell=1}^{k_O-1} \binom{k_O+\ell-1}{2\ell}_{-1} (-1)^{k_O\ell} 2^\ell q^\ell p^{k_O-\ell}$.

Then,

$$
\begin{aligned}
(p+dq)D_{k_O} = p &\left[\sum_{\ell=1}^{k_O-1} \binom{k_O+\ell-1}{2\ell}_{-1} (-1)^{k_O\ell} 2^\ell q^\ell p^{k_O-\ell}\right] \\
&+ q\left[\sum_{\ell=1}^{k_O-1} \binom{k_O+\ell-1}{2\ell}_{-1} (-1)^{k_O\ell} 2^\ell q^\ell p^{k_O-\ell}\right] \\
&- \left[\sum_{\ell=1}^{k_O-1} \binom{k_O+\ell-1}{2\ell}_{-1} (-1)^{k_O\ell} 2^\ell q^\ell p^{k_O-\ell}\right] q.
\end{aligned}
$$

From here we can split the summations into the odd and even ℓ values. Where an even $\ell = \ell_E$ and an odd $\ell = \ell_O$,

$$
\begin{aligned}
(p+dq)D_{k_O} = &\sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} (-1)^{(k_O\ell_E)} 2^{\ell_E} q^{\ell_E} p^{(k_O-\ell_E)+1} \\
&+ \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} - (-1)^{(k_O\ell_O)} 2^{\ell_O} q^{\ell_O} p^{(k_O-\ell_O)+1} \\
&+ \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} (-1)^{(k_O\ell_E)} 2^{\ell_E} q^{(\ell_E)+1} p^{k_O-\ell_E} \\
&+ \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} (-1)^{(k_O\ell_O)} 2^{\ell_O} q^{(\ell_O)+1} p^{k_O-\ell_O} \\
&- \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} - (-1)^{(k_O\ell_E)} 2^{\ell_E} q^{(\ell_E)+1} p^{k_O-\ell_E} \\
&- \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} (-1)^{(k_O\ell_O)} 2^{\ell_O} q^{(\ell_O)+1} p^{k_O-\ell_O}.
\end{aligned}
$$

It follows that the last two odd summations cancel and the (-1) values can be simplified such that,

$$
\begin{aligned}
(p+dq)D_{k_O} = &\sum_{\substack{\ell_E=2\\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_O-\ell_E)+1} \\
&+ \sum_{\substack{\ell_O=1\\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O-\ell_O)+1} \\
&+ \sum_{\substack{\ell_E=2\\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{(\ell_E)+1} p^{k_O-\ell_E} \\
&+ \sum_{\substack{\ell_E=2\\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{(\ell_E)+1} p^{k_O-\ell_E}.
\end{aligned}
$$

Now we can see that the last two even summations can be combined. Then,

$$
\begin{aligned}
(p+dq)D_{k_O} = &\sum_{\substack{\ell_E=2\\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_O-\ell_E)+1} \\
&+ \sum_{\substack{\ell_O=1\\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O-\ell_O)+1} \\
&+ \sum_{\substack{\ell_E=2\\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E+1} q^{(\ell_E)+1} p^{k_O-\ell_E}.
\end{aligned}
$$

Again to get D_{k+1} we add the term $dq(p^{k_O})$. However, in this case the term is equivalent to $qp^{k_O} - p^{k_O}q = qp^{k_O} - (-1)^{k_O}qp^{k_O} = 2qp^{k_O}$. So,

$$
\begin{aligned}
D_{k_O+1} = dq(p^{k_O}) + (p+dq)D_{k_O} = 2qp^{k_O} + &\sum_{\substack{\ell_E=2\\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_O+1)-\ell_E} \\
&+ \sum_{\substack{\ell_O=1\\ \ell_O \text{ is odd}}}^{k_O-2} \binom{k_O+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O-\ell_O)+1} \\
&+ \sum_{\substack{\ell_E=2\\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E+1} q^{(\ell_E)+1} p^{k_O-\ell_E}
\end{aligned}
$$

$$
\begin{aligned}
dq(p^{k_O}) + (p+dq)D_{k_O} &= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_O-1} \binom{k_O+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_O+1)-\ell_E} \qquad \text{(By Lemma 4.2)} \\
&\quad + \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O} \\
&= \sum_{\substack{\ell_E=2 \\ \ell_E \text{ is even}}}^{k_O-1} \binom{(k_O+1)+\ell_E-1}{2(\ell_E)}_{-1} 2^{\ell_E} q^{\ell_E} p^{(k_O+1)-\ell_E} \quad \text{(By Lemma 3.4)} \\
&\quad + \sum_{\substack{\ell_O=1 \\ \ell_O \text{ is odd}}}^{k_O} \binom{(k_O+1)+\ell_O-1}{2(\ell_O)}_{-1} 2^{\ell_O} q^{\ell_O} p^{(k_O+1)-\ell_O} \\
dq(p^k) + (p+dq)D_k &= \sum_{\ell=1}^{(k_O+1)-1} \binom{(k_O+1)+\ell-1}{2(\ell)}_{-1} (-1)^{(k_O+1)(\ell)} 2^{\ell} q^{\ell} p^{(k_O+1)-\ell}.
\end{aligned}
$$

Therefore when the theorem is true for odd k, then it is true for $k+1$. Hence,
$D_k = \displaystyle\sum_{\ell=1}^{k-1} \binom{k+\ell-1}{2\ell}_{-1} (-1)^{k\ell} 2^{\ell} q^{\ell} p^{k-\ell}$ for all $k \in \mathbb{N}$. □

When working with the binomial theorem, anticommutative associative algebra has proven itself to be a bridge between commutative associative algebra and the non-commutative associative algebra. Utilizing the relationship between the 1-binomial coefficient and the −1-binomial coefficient, we can derive an explicit alternative for Wyss's corollary for anticommutative values in associative algebras. Even though the theorem requires anticommutativity, it can be utilized in some noncommutative associative algebras, like quaternions and matrices, because in many of these algebras there exists a subsection of elements that anticommute.

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