Spring 2021

Game Chromatic Number on Segmented Caterpillars

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submitted in partial fulfillment of the requirements for Honors in Mathematics at the University of Mary Washington

Fredericksburg, Virginia

Spring 2021

This thesis by **Paige Beidelman** is accepted in its present form as satisfying the thesis requirement for Honors in Mathematics.

DATE	APPROVED
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5/5/2021	Keith Mellinger, Ph.D. committee member

Contents

Abstract

Graph theory is the study of sets of points (known as vertices) connected by lines (known as edges). The n-coloring game is a game played on a graph G with two players, Alice and Bob, such that they alternate to properly color vertices, meaning no adjacent vertices are the same color. Alice wins if every vertex is properly colored with n colors, otherwise Bob wins when a vertex cannot be colored using one of the avaliable colors. While strategies for winning this game may seem helpful, more interesting to us is the least number of colors needed for Alice to have a winning strategy, which is called the game chromatic number of G denoted $\chi_g(G)$. It is known that tree graphs have a game chromatic number of at most 4, but all the criteria for a tree T to have $\chi_g(T) = 4$ is unknown. To help in answering this question, we give the classification of a subclass of trees, which we call segmented caterpillar graphs. Caterpillars have at least one of each vertex of degree 2, 3, and 4, and therefore the game chromatic number cannot be determined by previous results.

1 Introduction

To acclimate the reader to graph theory, we begin in Section 1 with definitions of key terms about graphs and proceed on to define the coloring game. While competitive coloring is well researched, mathematicians have posed many open questions about the coloring game. Specifically, there are unsolved results of the game chromatic number of trees, which we define in Section 2. Then in Section 3, we then focus on a class of trees, known as caterpillars. This lead us to in Section 4 to define a new subclass of caterpillars, which we call segmented caterpillars. Building upon past results, we identified the game chromatic number for an infinite class of segmented caterpillars.

1.1 Graph Theory

Graph theory is the study of **graphs** G, which consists of sets of vertices and denoted $V(G)$, and sets of unordered pairs of vertices, known as edges and denoted as $E(G)$ [2]. Edges are depicted in illustrations as lines joining pairs of vertices, which are represented by points. There are many classes of graphs, but for this paper we focus on **simple graphs**. In a simple graph, each pair of vertices can have at most one edge join that pair of vertices. Vertices are **adjacent** if the vertices are connected by a single edge. The adjacent vertices of a vertex v can be called the **neighbors** of v.

Each vertex in a graph has a **degree**, which is the number of edges incident with that vertex. In a simple graph, the degree of vertex is equal to the number of neighbors. The maximum degree over all vertices in a graph G is denoted as $\Delta(G)$.

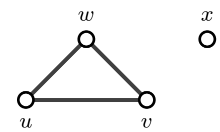

Figure 1: Graph G

For example, in Figure 1 the simple graph G has the set of vertices as $V(G) = \{u, v, w, x\}$ and set of edges $E(G) = \{uv, uw, vw\}$. Note for this paper, the graphs are undirected, meaning there is no direction on the edge between adjacent vertices. Thus in Figure 1, $uv = vu$ is true. Also, notice that vertex u is adjacent to v and w, but x does not have any neighbors. For this graph the $\Delta(G) = 2$ and the degree of x is 0.

Two graphs G and H are vertex disjoint if $V(G) \cap V(H) = \varnothing$ and edge disjoint if $E(G) \cap E(H) = \varnothing$. If G and H are both vertex disjoint and edge disjoint, then G and H are **disjoint**. The graph G' is a **subgraph** of G if $E(G') \subseteq E(G)$ and $V(G') \subseteq V(G)$ [2]. The subgraph G' is an **induced subgraph** if G' contains all edges $uv \in E(G)$ for $u, v \in V(G')$. For example in Figure 1, let H_1 and H_2 be subgraphs of G such that $V(H_1) = \{u, v, w\}$, $E(H_1) = \{uv, uw\}$, and $V(H_2) = \{x\}$. In G, only H_2 is an induced subgraph because H_2 contains all the original G edges between the vertices in the subgraphs. The subgraph H_1 does not contain the edge vw as seen in the graph G in Figure 1, so H_1 is not an induced subgraph. Also, H_1 and H_2 are disjoint since H_1 and H_2 do not share any edges or vertices.

A **path** P is a graph or subgraph with n distinct vertices $V(P) = \{v_1, v_2, \ldots, v_n\}$ and $n-1$ distinct edges $E(P) = \{v_1v_2, \ldots, v_{n-1}v_n\}$ [2]. The length of a path is its number of edges. In any graph, the **distance** from vertex v to vertex u is the length of the shortest path between the two vertices. **Connected** graphs have a path between every pair of vertices.

1.2 Coloring Game

Both edges and vertices can be colored in graphs either by physical colors shown on figures or through a function that assign labels to the graph. For our paper, we are only concerned with coloring vertices. For a graph G, we denote the coloring function $c : V(G) \to C$ such that C is the set of all colors. Typically in our paper, α represents red, β represents blue, γ represents yellow, and a gray vertex means the vertex is colored with one of the three colors in $C = \{\varnothing, \alpha, \beta, \gamma\}$. If $c(v) = \varnothing$, then vertex v is uncolored.

Vertices are **properly colored** if adjacent vertices v and u are colored differently such that $c(v) \neq c(u)$. The chromatic number of graph G is $\chi(G)$, which is the minimum number of colors required for every vertex in G to be properly colored.

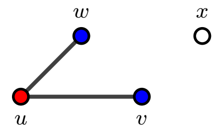

Figure 2: Graph G with $c(u) = \alpha$, $c(v) = c(w) = \beta$

Let us look to Figure 2 as an example where $c(u) = \alpha$, $c(w) = c(v) = \beta$, and $c(x) = \varnothing$. The adjacent vertices have different colors such that $c(v) \neq c(u)$ and $c(v) \neq c(w)$. Since u and w are not adjacent, it is admissible for $c(u) = c(w)$, but u and w do not necessarily need to have the same coloring. Notice the entire graph is not properly colored because $c(x) = \varnothing$, but we can clearly see $\chi(G) = 2$ by coloring x with either the α or β color.

The concept of a coloring game was first introduced by Martin Gardner in his Mathematical Games column in the *Scientific American* magazine in 1981 [4]. This is where Brams invented the map-coloring game with two players taking turns to color countries with n colors such that no countries sharing a border could be the same color. One player's goal is to have the map completely colored, while the other's goal is to the contrary of not having the graph properly colored with n colors [4]. Not until ten years later, when Bodlaender transferred the coloring game to graph theory, did this game gain any popularity in mathematics [4].

Definition 1.1 ([1])**.** The n**-coloring game** consists of two players, Alice and Bob, alternating turns to properly color vertices in a graph G with n colors. Alice wins if every vertex is properly colored, otherwise Bob wins when a vertex cannot be properly colored with n colors. In the n-coloring game, Alice plays first and neither player can skip or pass on their turn.

Example 1.2. Let us play the coloring game on the graph G as seen Figure 3.

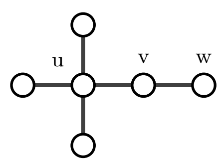

Figure 3: Graph G

Suppose first Alice colors u with color α as we see in Figure 4.

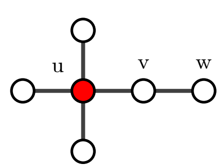

Figure 4: Graph G after Alice's turn

Then Bob colors vertex w with β as seen in Figure 5.

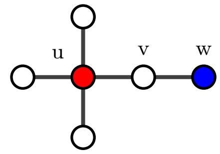

Figure 5: Graph G after Bob's turn

If Alice and Bob are playing the 2-coloring game, then Bob would win because vertex v can not be colored with α or β. However if Alice and Bob are playing 3-coloring game, then Alice can color v with γ as seen in Figure 6.

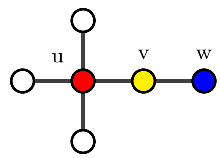

Figure 6: Graph G after Alice's turn

The Alice and Bob continue taking turns to properly color vertices. In this case Alice could eventually win the 3-coloring game because the remaining uncolored vertices can be properly colored with 3 colors.

While determining whether Alice or Bob wins is an exciting endeavor, we are more interested in the minimum number of colors needed to properly color the graph in the coloring game. Similar to the chromatic number, the **game chromatic number**, denoted as $\chi_g(G)$, is the least number of colors needed for Alice to have a winning strategy for the coloring game on a graph G [1]. For example in Figure 3, $\chi_g(G) = 3$ because regardless of Bob's moves, Alice has a winning strategy with 3 colors, but not 2 colors.

It is important to note that the maximum degree and the chromatic number are bounds for the game chromatic number. The vertex v with maximum degree will have the most neighbors, so there is a possibility of each neighbor having a unique color including v. Hence $\Delta(G)+1$ is the upper bound for the game chromatic number. In order for the Alice to have a winning strategy, she must use at least as many colors needed to properly color a graph without playing the coloring game. Remember that the chromatic number of a graph is the minimum number of colors needed to properly color a graph by Brooks' Theorem. Similarly Bodlaender proved

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$$

for a graph G [1]. Let us look at an example that meets lower and upper bounds for the game chromatic number. One example is a complete graph, which has every vertex adjacent to all other vertices. In Figure 7, the complete graph has $\chi(G) = 5$. Also, we see $\Delta(G) = 4$, so the upper bound is also 5. Thus $\chi_g(G) = 5$ for the graph in Figure 7. Note that all complete graphs have a game chromatic number equal to the chromatic number.

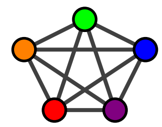

Figure 7: Complete Graph G

2 Game Chromatic Number on Trees

We now investigate the game chromatic number of a specific class of graphs, known as tree graphs. To understand trees, we need to define a cycle. A graph G contains a **cycle** if G has a path $P = x_0, x_1, \ldots, x_n$ such that $x_1 x_n$ is an edge in G. If starting from one vertex we travel or *walk* along a path and get back to our starting vertex, then the graph contains a cycle. Note that graphs can have more than one cycle.

Definition 2.1. **Tree** graphs are connected simple graphs that do not contain any cycles.

For example, Figure 1 is not a tree because the path $w \to v \to u \to w$ is a cycle in G and the graph is disconnected. However, Figure 2 and Figure 3 are trees because both are acyclic and connected simple graphs.

We can also expand the idea of trees to disjoint graphs. A forest is a disconnected graph such that each connection component is a tree. Continuing with the tree metaphor, we label the vertices of degree 1 as **leaves**. All non-leave vertices, known a central vertices of a tree, have degree at least 2.

2.1 Trunking

By breaking a tree apart as Alice and Bob play, trunking is a tool for proving Alice or Bob win on a tree. This trunking method was formed in [3], but a similar trunking method has been used for the coloring game dating back to Faigle [5].

Definition 2.2. A **trunk** T of graph G is a maximal connected subgraph of G such that every colored vertex in T is a leaf of T.

The set of all trunks is $R(G)$ with each uncolored vertex in exactly one trunk and each colored vertex v in $deg(v)$ many trunks. We can define the trunking method iterative as

Alice and Bob color vertices. Once a vertex u in a trunk T is colored, then the trunk is divided into $n = deg(u)$ trunks at u. Now we have $n - 1$ more trunks in $R(G)$ because we replaced T with n trunks.

Let us run through an example of using the trunking method on graph G. At first the entire graph G is the trunk because there are no colored leaves.

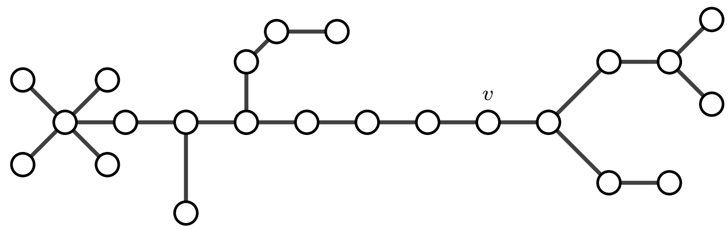

Figure 8: Tree G is the trunk.

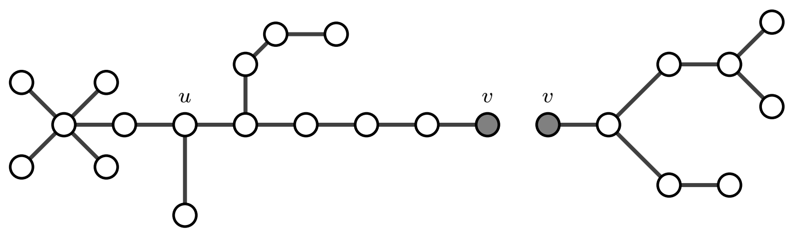

Figure 9: After the vertex v in G is colored we can divide G into two trunks T_1 and T_1, since $deg(v) = 2$.

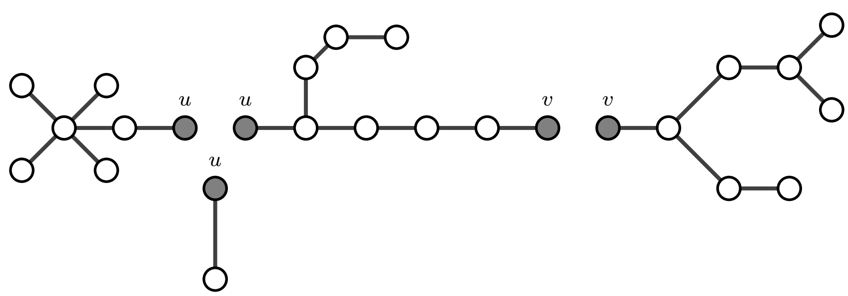

Figure 10: Vertex u is colored in T_1, so T_1 is now broken into 3 trunks. Thus $R(G)$ has 4 trunks.

In Section 1.2, the n-coloring game is defined by Alice coloring first and neither player can pass on their turn. However we will now define two variations of the n-coloring game as designed by Dunn et al. [3].

Definition 2.3. In the **n-modified coloring game**, denoted as n-MCG, Bob plays first on a partially colored graph. Then, Alice and Bob alternate properly coloring vertices, but

Bob can skip his turn. Alice wins if every vertex is properly colored with n colors, otherwise Bob wins.

Lemma 2.4. *[3] If Alice can win the n-MCG on every trunk in a partially colored forest F, she can win the n-coloring game on F.*

Lemma 2.5. *[3] If T is a trunk with at most one colored vertex and at most 7 vertices, Alice can win the 3-MCG on T.*

The purpose of the modified coloring game is to help prove Alice could win with n colors as seen in Lemma 2.4. Thus the modified coloring game is instrumental to proving n as the lower bound of the game chromatic number.

Similarly, [3] designed the expanded coloring games to demonstrate Bob's winning strategy with k colors. Therefore the expanded coloring game is helpful with proving n as the lower bound for the game chromatic number. In the **n-expanded coloring game**, denoted as n-ECG, Alice plays first on a partially colored graph. For Alice's turns she can color a vertex, add a colored leaf to any vertex, or pass her turn. Otherwise everything in the n-coloring game applies to the n-ECG variation.

Lemma 2.6. *[3] Let F be a partially colored forest and let F' be an induced connected subgraph of F. If Bob can win the 3-ECG on F' and every vertex in F' does not have a colored neighbor in F, then Bob can win the 3-coloring game on F.*

In particular, these lemmas with trunking method, MCG, and ECG will be helpful in proving Alice wins, which in turns proves the upper bound of the game chromatic number of a graph.

2.2 Past Results for Trees

As a first result, Bodlaender [1] proved that Alice could always win the 5-coloring game on any tree. Yet, Bodlaender was never able to find a graph with the game chromatic number of 5. Thus in [5], the upper bound was improved by proving trees have a game chromatic number of at most 4. This is a strict upper bound because [3] found trees with game chromatic number equal to 4. We also know that any connected graph with at least 2 vertices will have a game chromatic number of at least 2. In [3], the conditions for a forest F to have $\chi_g(F) = 2$ are stated in Theorem 2.7.

Theorem 2.7 (Dunn et al. [3])**.** *Let F be a forest with the longest path having length ℓ. Then $\chi_g(F) = 2$ if and only if*

- $1 \leq \ell \leq 2$ *or*
- $\ell = 3$*, the number of vertices is odd, and every tree in the forest with a distance 3 is a path.*

We can also identity the game chromatic number based on the number vertices in a forest. If F is a forest with $n \leq 13$ vertices, then $\chi_g(F) = 3$ [3]. However the requirements for whether a forest or tree having a game chromatic number of 3 or 4 are still unknown according to [4]. In this paper, we prove results on the differentiation of the game chromatic number of trees to help answer this open question.

Question: What is the criteria for differentiating between trees with game chromatic number of 3 and 4? [4]

3 Game Chromatic Number on Caterpillars

Instead of studying the entire class of trees, we will focus one subclass of trees. There are many classes of connected graphs without cycles, such as lobster graphs, polygraphs, and leaf powers. In order to help answer a portion of the above question, we will investigate when the game chromatic number is 3 or 4 in a subclass of trees called caterpillars. A **caterpillar** graph, denoted as $cat\{k_1, k_2, \ldots, k_n\}$, is a tree with n central vertices such that each central vertex v_i has k_i leaves. For example, in Figure 11 the caterpillar C contains 8 central vertices, which are circled in red.

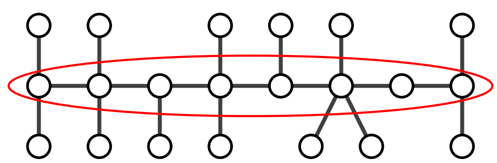

Figure 11: Caterpillar $C = cat\{2, 2, 1, 2, 1, 3, 0, 2\}$

3.1 Past Results for Caterpillars

The most trivial set of caterpillars are paths, which is $P_n = cat\{0, 0, 0, \ldots, 0\}$ with $1 \leq n$ vertices. The upper bound is $\chi_g(P_n) \leq \Delta(P_n) + 1 = 3$. Now let's play the 2-coloring game on P_n with $n \geq 4$ vertices. After Alice's first turn, Bob can always color a vertex a distance 2 away with a new color. Thus an uncolored vertex is adjacent to two different colors, so a third color is needed. We can see an example of the 2-coloring game on a path P_4 in Figure 12, where v can not be colored with 2 colors. Since Bob can win the 2-coloring game, then $\chi_g(P_n) = 3$ for $n \geq 4$.

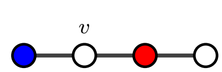

Figure 12: Path P_n after Bob's turn

Another trivial caterpillar is a star, which is $cat\{n\}$ with $n \geq 1$. In the 2-coloring game, Alice can play first, so Alice colors the only central vertex with color α. Then all the leaves can be properly colored with β. Therefore the star has a game chromatic number of 2. In Figure 13, Alice has just played on the star $cat\{8\}$.

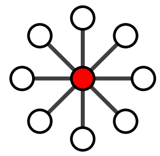

Figure 13: Star with 8 leaves after Alice's first turn

To solve the problem of game chromatic number of caterpillars, Furtado et al used maximum degree and graphs without vertices of a certain degree as the criteria for identifying the game chromatic number. Other than the game chromatic number of trivial caterpillar cases, more general results of the game chromatic number were researched by Furtado et al. Below is the list the major results in [7] and [6].

- A caterpillar G with less than 4 vertices of degree 4 has $\chi_g(G) = 3$.
- Any caterpillar G with $\Delta(G) = 3$ has $\chi_g(G) \leq 3$.
- A caterpillar G without a vertex of degree 2 has $\chi_g(G) = 4$ if and only if G has at least 4 vertices with degree 4 or greater.
- A caterpillar G without a vertex of degree 3 has $\chi_g(G) = 4$ if and only if G has an induced subgraph of the caterpillar of family Q from [7].

With only 14 vertices $G = cat\{0, 2, 2, 2, 2, 0\}$ in Figure 14 is the smallest tree, which happens to be a caterpillar, with game chromatic number of 4. However, a tree with game chromatic number of 4 does not need to contain G as a subgraph, as we will see in Section 4.

Theorem 3.1. *[3] If G is as seen in Figure 14, then Bob can win the 3-ECG on G. Moreover, G is a minimal example of a tree with $\chi_g(G) = 4$.*

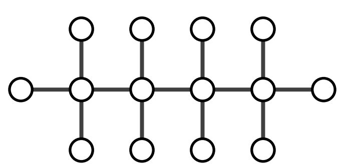

Figure 14: Smallest tree G with $\chi_g(G) = 4$

The previous results classify multiple caterpillars, yet caterpillars with more than 13 vertices that contain a vertex of degree 2, degree 3, and at least 4 vertices of at least 4 degree has not been classified. Thus we created a new subclass of caterpillars, called *segmented caterpillars*.

4 Game Chromatic Number on Segmented Caterpillars

To guarantee a caterpillar with a vertex of degree 2, we created a segmented caterpillar. Then with additional restrictions on the segments, the caterpillar will have a degree 3 and at least 4 vertices of at least 4 degrees. A **segment** is a sequence $S = \{k_1, \ldots, k_n\}$ such that $k_i \geq 1$ for $0 \leq i \leq n$ and $n \geq 1$. A **segmented caterpillar** is a caterpillar $H = cat\{S_1, 0, S_2, 0, S_3, 0, \ldots, 0, S_m\}$ with m segments $\{S_1, S_2, \ldots, S_m\}$. Note that this notation of caterpillars was used in [6] and [7] and we extended the notation for segments. In this paper, we will work with **uniform** segmented caterpillars, which occur when segments are all the same.

In this section, we consider uniform segmented caterpillars with segments of $\{2\}, \{2, 2\}$, or $\{2, 2, 2\}$. Notice in the segmented caterpillar, the first vertex has exactly 2 leaves, so the graph has a vertex of degree 3. For the game chromatic number of the segmented caterpillar to be non-trivial by results of Dunn et al. [3] and Furtado et al. [7], we assume the segmented caterpillar has more than 13 vertices and at least 4 vertices of degree at least 4. The first case to consider is when each segment is $\{2\}$.

4.1 Segments of $\{2\}$

Let us investigate uniform segmented caterpillars with segments of $\{2\}$. To begin, we need to consider partially colored trunks that could appear while playing the n-coloring game. To prove the game chromatic number of all uniform segmented $\{2\}$ caterpillars is 3 as seen below in Theorem 4.5, we will to prove in Lemma 4.3 and Lemma 4.4 that Alice can win on all possible trunks.

In order to gain a better understanding of the criteria for a tree having a game chromatic number of 3 and 4, the reader should be knowledgeable of dangerous vertices. A **dangerous** vertex v has at most as many uncolored adjacent vertices as legal colors available to color v in the n-coloring game[3]. Note a dangerous vertex can have colored neighbors or no colored neighbors. In an n-coloring game, a **very dangerous vertex** has $n - 1$ uniquely colored neighbors and at least one uncolored neighbor that can be properly colored with the n^{th} color. We notice that all very dangerous vertices are dangerous vertices, but not vice versa. To see the difference between dangerous vertices and very dangerous vertices, let us look at Figure 15, which contains three graphs in the 3-coloring game. Since u_1 has 2 possible colors in the 3-coloring game and 2 uncolored neighbors, then u_1 is a dangerous vertex. The vertex u_2 is a very dangerous vertex because it has 2 uniquely colored neighbors and an uncolored neighbor.

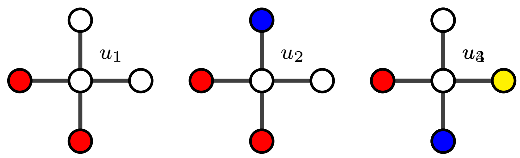

Figure 15: In the 3-coloring game, u_1, u_2, and u_3 are dangerous vertices, u_2 is a very dangerous vertex, and u_4 can not be colored with 3-coloring game. Thus Alice wants to avoid 3 unique colored leaf adjacent to one vertex.

In order to prove more general statements about the game chromatic number of trees, we utilize the concepts of dangerous and very dangerous vertices.

Lemma 4.1. *If a graph G does not have any dangerous vertices or G has exactly one dangerous vertex with the same colored neighbors, then Alice wins the 3-MCG on G.*

Proof. Let G be a graph where Alice and Bob are playing the 3-modified coloring game. Thus we will prove that Alice wins for each move Bob makes in his turn. Since Bob and Alice are playing the 3-MCG, Bob plays first. Trivially, if G has no dangerous vertices, then Bob can color any vertex because every vertex in G has more legal colors available than uncolored adjacent vertices. Thus every vertex can be properly colored, so Alice will win the 3-MCG on G.

Assume G has exactly one dangerous vertex v with the colored neighbors having the color α. If Bob colors v, Alice colors any vertex. If Bob colors an adjacent vertex to v with α, then Alice colors v with β. If Bob passes or colors a vertex not adjacent to v, then Alice colors v. Thus after Bob's turn, Alice can eliminate any dangerous vertices in T. Therefore Alice will win the 3-MCG on G because every vertex can be properly colored.

□

Lemma 4.2. *If the graph G has at least two very dangerous vertices that are not adjacent, then Bob wins the* 3*-coloring game on G.*

Proof. Suppose a graph G has two non-adjacent very dangerous vertices v and u. In the 3-coloring game, Alice plays first. If Alice does not color u, then Bob colors a leaf of u with a new third color. Thus u cannot be colored in the 3-coloring game, so Bob wins.

If Alice colors the very dangerous vertex u, then Bob colors a leaf of v with a new third color. Thus v cannot be colored in the 3-coloring game, so Bob wins. □

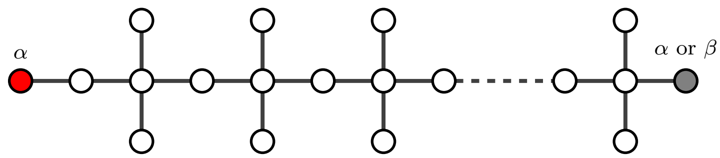

Figure 16: Caterpillar $H_1 = cat\{0, 0, S_1, 0S_2, 0, S_3, 0, \ldots, 0, S_n, 0\}$ with uncolored $S_i = \{2\}$ for $1 < i < n$.

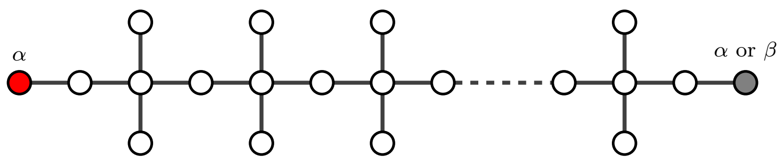

Figure 17: Caterpillar $H_2 = cat\{0, 0, S_1, 0S_2, 0, S_3, 0, \ldots, 0, S_m, 0, 0\}$ with uncolored $S_j = \{2\}$ for $1 < j < m$.

Lemma 4.3. *If a partially colored segmented caterpillar G is H_1 as seen in Figure 16 or H_2 as seen in Figure 17, then Alice wins the 3-MCG on G.*

Proof. Through induction on the number of dangerous vertices, we will prove that Alice wins the 3-MCG on G, which is the partially colored graph H_1 as seen in Figure 16 or H_2 as seen in Figure 17. If G has only one dangerous vertex, then Alice wins with 3 colors by Lemma 4.1.

Assume Alice wins the 3-MCG on H_1 and H_2 with each having n dangerous vertices. We will prove Alice wins the 3-MCG for the partially colored graph H_1 and H_2 with $n+1$ dangerous vertices.

First, let G have the structure of H_1 with $n+1$ dangerous vertices. Since we are playing the 3-MCG, Bob plays first by either passing, coloring a dangerous vertex, degree 2 vertex, or a leaf.

- If Bob passes on his turn, then Alice colors the dangerous vertex of S_{n+1}. Thus G has the structure of H_2 with n dangerous vertices and other trunks with no dangerous vertices.
- If Bob colors a dangerous vertex, then Alice colors any dangerous vertex. Thus G has one or two trunk(s) of the form H_2, possibly one trunk of the form H_1 with each trunk having at the most n dangerous vertices, and other trunks with no dangerous vertices.
- If Bob colors a vertex of degree 2, then Alice colors the dangerous vertex of S_{n+1}. Thus G has one or two trunk(s) of H_1 each with at most n dangerous vertices and other trunks with no dangerous vertices.

- If Bob colors a leaf x, then Alice colors the dangerous vertex adjacent to x. Now G has one trunk of H_2 with at most n dangerous vertices, possibly one trunk of H_1 with at most n dangerous vertices, and other trunks with no dangerous vertices.

Let G have the structure of H_2 with $n+1$ dangerous vertices. Since we are playing the 3-MCG, Bob plays first.

- If Bob passes or colors a dangerous vertex, then Alice colors any dangerous vertex. Thus G has at most three trunks with the structure of H_2 each with at most n dangerous vertices and other trunks with no dangerous vertices.
- If Bob colors a vertex x of degree 2, then Alice colors either dangerous vertex adjacent to x. Thus G' has at most one trunk of H_1 with at most n dangerous vertices, one trunk of H_2 with at most n dangerous vertices, and other trunks with no dangerous vertices.
- If Bob colors a leaf u, then Alice colors the dangerous vertex adjacent to u. Thus G has at most two H_2 with at most n dangerous vertices, and other trunks with no dangerous vertices.

Alice can win the 3-MCG on any trunks without any dangerous vertices by Lemma 4.1. Thus we only focus on the trunks with more than one dangerous vertex. By the inductive hypothesis and Lemma 2.4, Alice wins the 3-MCG on G. □

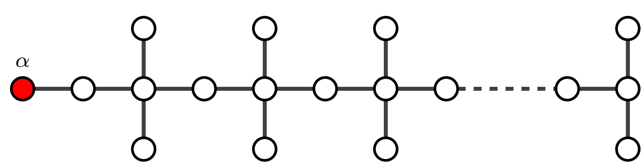

Figure 18: Segmented caterpillar $H = cat\{0, 0, S_1, 0, S_2, 0, S_3, 0, \dots, 0, S_n\}$ with uncolored $S_i = \{2\}$ for $1 < i \leq n$.

Lemma 4.4. *If H is the partially colored segmented caterpillar as seen in Figure 18, then Alice wins the 3-MCG on H.*

Proof. Let H be the partially colored segmented caterpillar as seen in Figure 18. If H has no dangerous vertices, then Alice wins the 3-MCG by Lemma 4.1.

Assume Alice wins the 3-MCG on H with n dangerous vertices. We will prove Alice then wins the 3-MCG on H' with $n+1$ dangerous vertices. Since we are playing the 3-MCG, then Bob plays first by passing, coloring a dangerous vertex, a vertex of degree 2, or a leaf.

- If Bob passes, then Alice colors the dangerous vertex of S_2. Thus H' has a trunk of H with n dangerous vertices and other trunks with no dangerous vertices.
- If Bob colors a dangerous vertex, then Alice colors a dangerous vertex. Thus H' has a trunk of H with at most n vertices, two trunks of H_2 as seen Figure 17, and other trunks with no dangerous vertices.
- If Bob colors a vertex of degree 2, then Alice colors the adjacent dangerous vertex closer in distance to S_{n+1}. Thus H' has a trunk of H with n dangerous vertices, possibly a trunk of H_1 as seen Figure 16, and other trunks with no dangerous vertices.
- If Bob colors a leaf, then Alice colors the adjacent dangerous vertex. Thus H' has possibly a trunk of H with at most n dangerous vertices, a trunk of H_1 as seen in Figure 16, and other trunks with no dangerous vertices.

By Lemma 4.1, Alice can win the 3-MCG on any trunks without any dangerous vertices. Thus we only focus on the trunks with more than one dangerous vertex. By the inductive hypothesis and Lemma 4.3, Alice wins the 3-MCG on each trunk in H'. Thus by Lemma 2.4 Alice wins the 3-MCG on H' with $n+1$ dangerous vertices. □

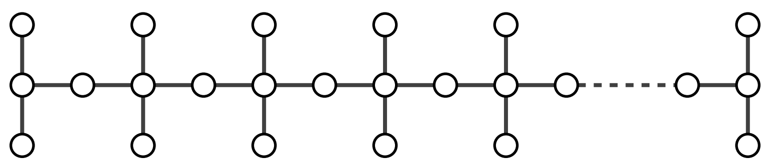

Figure 19: Segmented $\{2\}$ Caterpillar

Theorem 4.5. *If G is a uniform segmented $\{2\}$ caterpillar as seen in Figure 19, then $\chi_g(G) = 3$.*

Proof. Let Alice and Bob play the 3-coloring game on $G = cat\{S_1, 0, S_2, 0, \ldots, 0, S_n\}$ with $S_i = \{2\}$ for $1 \geq i \geq n$. Alice goes first by coloring one of the vertices of degree 3. Alice can win the 3-MCG of each trunk by Lemma 4.4 and Lemma 4.1. Thus Alice wins the 3-coloring game by Lemma 2.4.

Alice can not win the 2-coloring game by Theorem 2.7. Therefore, Alice has a winning strategy with 3 colors, so $\chi_g(G) = 3$. □

The uniform segmented $\{2\}$ caterpillar has a game chromatic number of 3, which is non-trivial because the caterpillar contains a degree 3 vertex, degree 2 vertex, more than 4 vertices with degree at least 4, and more than 13 vertices. We have classified uniform segmented $\{2\}$ caterpillars, so the next extension is the game chromatic number of uniform segmented caterpillars with segments of $\{2, 2\}$.

4.2 Segments of $\{2,2\}$

To prove that a uniform segmented $\{2,2\}$ caterpillar has a game chromatic number of 4, we will use a partially colored trunk that [3] proved Bob to win the 3-extended coloring game.

Lemma 4.6 (Dunn et al. [3])**.** *Let G be the partially colored tree shown in Figure 20 that may have an additional leaf colored β not shown in the figure. Then Bob can win the 3-ECG on G.*

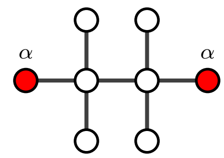

Figure 20: By Lemma 4.6, Bob wins the 3-coloring game.

Lemma 4.7. *If a segmented caterpillar G contains two disjoint subgraphs that are each $cat\{0,2,2,0,2,2,0\}$, then $\chi_g(G) = 4$.*

Proof. Let G be a segmented caterpillar such that G contains at least two disjoint subgraphs $cat\{0,2,2,0,2,2,0\}$, called H_1 and H_2. Alice colors any vertex x with α. Since $H_1 \cap H_2 = \varnothing$, then x is either in H_1 or H_2, but not both. Without loss of generality, assume $x \notin H_1$, so $v \in V(H_1)$ is an uncolored vertex of degree 2 as seen in Figure 21. Bob colors v with color β. Now H_1 has two isomorphic subgraphs $H_1' = H_1'' = cat\{0,1,1,0\}$ with $H_1' \cap H_1'' = v$. Without loss of generality, Alice colors a vertex $y \notin H_1'$, so Bob colors the vertex without leaves in H_1' with color β. By Lemma 4.6, Bob can win 3-ECG on H_1'. By Lemma 2.6, Bob wins the 3-coloring game on G. Since $\chi_g(G) \leq 4$ and $\chi_g(G) > 3$, then $\chi_g(G) = 4$. □

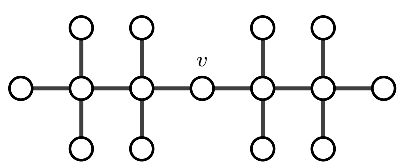

Figure 21: Subgraph $H_1 = H_1' \cup H_1''$

Lemma 4.8. *If a segmented caterpillar G contains a subgraph equal to one $cat\{0,2,2,0,2,2,0,2,2,0\}$, then $\chi_g(G) = 4$.*

Proof. Let H be the subgraph $cat\{0,2,2,0,2,2,0,2,2,0\}$ as seen in Figure 22. We will prove Bob can win on this subgraph for every possible turn Alice takes by proving Bob wins the 3-ECG.

Case 1: Alice passes her first turn. Bob colors vertex y with α. If Alice colors a vertex in segment 2, then Bob colors z with α. Otherwise, if Alice does not color a vertex in segment 2, Bob colors x with α. Thus Bob has created the subgraph in Figure 20.

Case 2: Without loss of generality between coloring in segments 1 and 3, Alice colors a vertex in segment 1 with α. Bob colors y with β. If Alice colors z or a vertex in segment 3, then Bob colors x with β. Otherwise if Alice passes, colors x, w, a vertex in segment 1, or a vertex in segment 2, then Bob colors z with β. Thus Bob has created the subgraph in Figure 20.

Case 3: Without loss of generality between coloring the vertex adjacent to x or to y in segment 2, Alice colors with α a dangerous vertex that is adjacent to x or a leaf of the dangerous vertex that is adjacent to y. With β, Bob colors y. If Alice colors the very dangerous vertex, then Bob colors z with β. Thus Bob has created the subgraph in Figure 20. Otherwise if Alice does not color the very dangerous vertex, then Bob colors the leaf of the very dangerous vertex with γ. Thus, the very dangerous vertex would need a 4th color.

Case 4: Alice colors a vertex of degree 2 with α. Bob then colors the vertex of degree 2 that is distance 3 away. Thus Bob has created the subgraph in Figure 20.

Since in every case Bob has created the subgraph in Figure 20 or created a vertex that requires a 4th color to be properly colored, Bob wins the 3-ECG on the subgraph H. Thus by Lemma 2.6, Bob wins the 3-coloring game. Since $\chi_g(G) \leq 4$ and $\chi_g(G) > 3$, then $\chi_g(G) = 4$. □

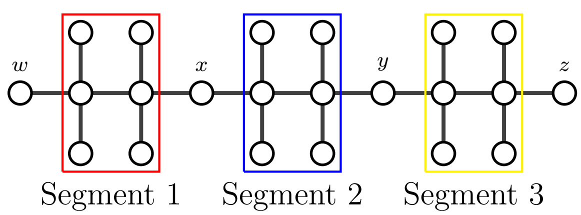

Figure 22: Caterpillar $H = cat\{0, S_1, 0, S_2, 0, S_3, 0\}$ with $S_1 = S_2 = S_3 = \{2, 2\}$

Lemma 4.7 and Lemma 4.8 can be extended to non-uniform segmented caterpillars, but we classify the uniform segmented $\{2, 2\}$ caterpillars that have two disjoint subgraphs of $cat\{0, 2, 2, 0, 2, 2, 0\}$ or a subgraph of $cat\{0, 2, 2, 0, 2, 2, 0, 2, 2, 0\}$.

Lemma 4.9. *If a graph G as seen in Figure 23, then Alice wins the* 3*-MCG on G.*

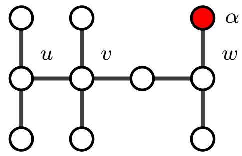

Figure 23: Partially colored caterpillar $G = cat\{2, 2, 0, 2\}$ such that u can have more uncolored leaves, w can have more uncolored or α colored leaves, or w is colored.

Proof. Suppose Alice and Bob are playing the 3-MCG on a caterpillar graph G as seen in Figure 23.

Case 1: If Bob colors a dangerous vertex v, then there are two trunks with vertex v beginning in both trunks. Since each trunk contains exactly one dangerous vertex each, Alice will win the 3-MCG on each trunk by Lemma 4.1.

Case 2: If Bob colors a leaf of u with α, then Alice colors v with α. Since there are two trunks with exactly one dangerous vertex each, then Alice wins the 3-MCG on each trunk by Lemma 4.1.

Case 3: If w is uncolored and Bob colors a vertex adjacent to w, then Alice colors w with β or γ.

- If Bob colors a vertex adjacent to v with α, Alice colors v with β or γ. With only one dangerous vertex u, Alice will win the 3-MCG by Lemma 4.1.
- If Bob does not color a vertex adjacent to v, then Alice colors u. Alice wins the 3-MCG on T by Lemma 4.1 as only v is uncolored with a single colored neighbor.

Case 4: If Bob does not color anything mentioned in Cases 1-3, Alice colors v to create two trunks. Since two trunks each having only one dangerous vertex, then Alice wins the 3-MCG on each trunk by Lemma 4.1.

Alice wins on each trunk, so by Lemma 2.4 Alice wins on G. □

Lemma 4.10. *If G is the partially colored caterpillar graph as seen in Figure 24, then Alice can win the 3-MCG on G.*

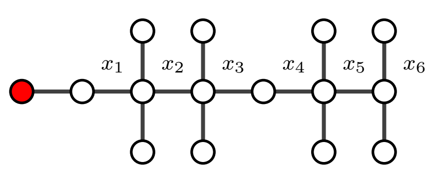

Figure 24: Partially colored caterpillar $G = cat\{0, 0, 2, 2, 0, 2, 2\}$.

Proof. Let G be the partially colored caterpillar graph as seen in Figure 24. Bob can play first.

- If Bob colors x_4 or leaf of x_3 with β, then Alice colors x_2 with β, so G has two trunks. By Lemma 4.9 and Lemma 4.1, Alice wins on both trunks.
- If Bob colors x_5 or x_6 with β, then Alice colors x_3 with β, so G has 3 trunks each with only one dangerous vertex. By Lemma 4.9, Alice wins on each trunk.
- If Bob colors a leaf of x_5 with β, then Alice colors x_6 with β, so G has one trunk, which Alice wins on by Lemma 4.1.
- If Bob colors a leaf of x_6 with β, then Alice colors x_6 with γ, so G has one trunk, which Alice wins on by Lemma 4.1.
- If Bob colors x_2, so G has 2 trunks, then Alice can win on by Lemma 4.9 and Lemma 4.1.
- If Bob colors a vertex not mentioned in the above cases or Bob passes, then Alice colors x_2, so G has 2 trunks. By Lemma 4.9 and Lemma 4.1, Alice wins on both trunks.

Alice wins on every trunk of G. Thus by Lemma 2.4, Alice wins the 3-MCG on G and $\chi_g(G) > 2$ by Theorem 2.7. □

Theorem 4.11. *If a uniform segmented caterpillar G has less than 5 segments of $\{2, 2\}$, then $\chi_g(G) = 3$.*

Proof. Let G be a uniform segmented caterpillar of less than 5 segments of $\{2, 2\}$.

- If G has 2, 3, or 4 segments, Alice colors the dangerous vertex that is distance 4 from one of the vertices of degree 3. Thus G has a trunk defined by Figure 24, possibly a trunk defined by Figure 23, and two trivial trunks without any dangerous vertices. Thus by Lemma 4.9 and Lemma 4.10, Alice wins the 3-MCG on each trunk. By Lemma 2.4, Alice wins the 3-coloring game on G.
- If G has 1 segment, then Alice colors a dangerous vertex. Thus G has only one dangerous vertex, so Alice wins the 3-coloring game by Lemma 4.1.

Therefore the $\chi_g(G) = 3$, since Alice is able to win the 3-coloring game on G. □

Theorem 4.12. *A uniform segmented $\{2, 2\}$ caterpillar G has $\chi_g(G) = 4$ if and only if G has at least 5 segments.*

Proof. Let G be a uniform segmented $\{2,2\}$ caterpillar having at least 5 segments of $\{2,2\}$. Thus G will have the subgraph $cat\{0,2,2,0,2,2,0,2,2,0\}$ as defined by Figure 22. Therefore $\chi_g(G) = 4$.

Suppose G has $\chi_g(G) = 4$ and has less than 5 segments. By Theorem 4.11, G has $\chi_g(G) = 3$ because G has less than 5 segments of $\{2,2\}$. This is a contradiction. Therefore G must have at least 5 segments. □

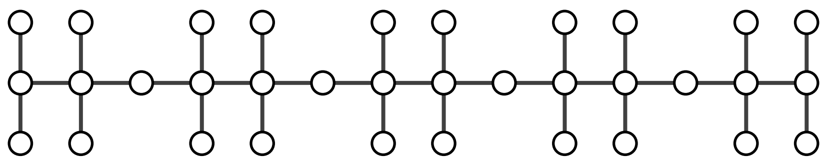

Figure 25: Segmented Caterpillar with 5 segments of $\{2,2\}$.

4.3 Segments of $\{2,2,2\}$

Similar to uniform segmented $\{2,2\}$ caterpillars, the game chromatic number of uniform $\{2,2,2\}$ segmented caterpillars stems from Lemma 4.7.

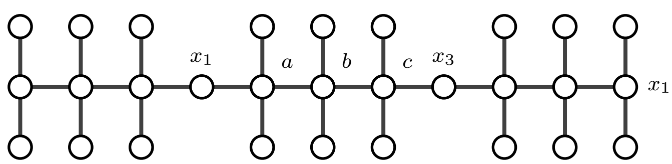

Figure 26: $\chi_g(G) = 4$

Lemma 4.13. *If G is a uniform segmented $\{2,2,2\}$ caterpillar with 3 segments as seen in Figure 26, then Bob wins the 3-ECG on G.*

Proof. Let G be a uniform segmented $\{2,2,2\}$ caterpillar with 3 segments as seen in Figure 26. In the ECG Alice is the first to play and Alice can skip.

Case 1: If Alice colors a or a leaf of b with α, then Bob colors x_2 with α. Thus G contains the subgraph as defined by Figure 20. By Lemma 4.6, Bob wins the 3-ECG.

Case 2: If Alice colors b with α, then Bob colors x_2 with β.

- If Alice colors c with γ, then Bob colors x_3. Thus G contains the subgraph as defined by Figure 20. By Lemma 4.6, Bob wins the 3-ECG.

- If Alice does not color c, then Bob colors the leaf of c with γ. Thus c would require a 4th color. Thus Bob wins the 3-ECG on G.

Case 3: If Alice colors c with α, then Bob colors x_1. Thus G contains the subgraph as defined by Figure 20. By Lemma 4.6, Bob wins the 3-ECG.

Case 4: If Alice does not color the vertices mentioned in Cases 1-3, then G contains $cat\{0, 2, 2, 0, 2, 2, 0\}$ and it is Bob's turn. As shown in the proof of Lemma 4.7, Bob will win the 3-ECG. □

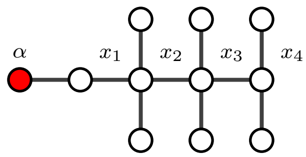

Figure 27: $G = cat\{0, 0, 2, 2, 2\}$

Lemma 4.14. *Alice wins the 3-MCG on $G = cat\{0, 0, 2, 2, 2\}$ as seen in Figure 27.*

Proof. Let $G = cat\{0, 0, 2, 2, 2\}$ as seen in Figure 27. Bob plays first. If Bob colors x_2, then Alice colors x_3. If Bob colors x_4 or leaf of x_3 with α, then Alice colors x_2 with α. If Bob colors a leaf of x_4 with α, then Alice colors x_3 with β. If Bob colors anything not mentioned or passes, Alice colors x_2. Alice wins the 3-MCG on the each trunk by Lemma 2.5 or Lemma 4.1. □

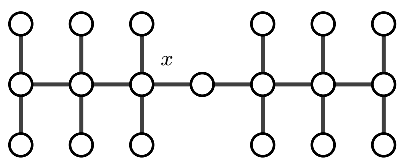

Figure 28: Uniform segmented caterpillar G

Lemma 4.15. *If G is a uniform segmented $\{2, 2, 2\}$ caterpillar with 1 or 2 segments, then $\chi_g(G) = 3$.*

Proof. If G has 2 segments, then Alice plays first and colors x as seen in Figure 28. If G has 1 segment, then Alice colors either vertex of degree 3. Thus Alice wins the 3-MCG on each trunk by Lemma 2.5 and Lemma 4.14. Since Alice wins the 3-MCG on G, then $\chi_g(G) = 3$. □

Theorem 4.16. *A uniform segmented $\{2, 2, 2\}$ caterpillar G has $\chi_g(G) = 4$ if and only if G has at least 3 segments.*

Proof. Let G be a uniform segmented $\{2,2,2\}$ caterpillar with at least 3 segments. If G has exactly 3 segments, then Bob wins the 3-ECG by Lemma 4.13. If G has more than 3 segments, then Bob wins the 3-ECG by Lemma 4.7. Since Bob wins the 3-ECG, then by Lemma 2.6, $\chi_g(G) = 4$.

Suppose G has $\chi_g(G) = 4$ and has less than 3 segments. By Lemma 4.15, G has $\chi_g(G) = 3$ because G has less than 3 segments of $\{2,2,2\}$. This is a contradiction. Therefore G must have at least 3 segments. $\square$

The next largest uniformed segmented caterpillar has segments of $cat = \{2,2,2,2\}$. Any segments with more central vertices than $cat = \{2,2,2,2\}$ would have a game chromatic number of 4 for the same reasoning.

5 Conclusion

Posed with the question of identifying the difference in trees with game chromatic number 3 or 4, we focus on classifying the game chromatic number of a new subclass of caterpillars. We were able to find the criteria for non-trivial uniform segmented caterpillars. Our next questions and directions for investigation include:

- What is the game chromatic number of a forest such that each connected component is a segmented caterpillar with segments of $\{2\}$?
- What is the criteria of non-uniform segmented caterpillars to have game chromatic number of 3 or 4?
- Can we extend the results of segmented caterpillars with segments of $\{2\}$, $\{2,2\}$, $\{2,2,2\}$ to lobster graphs, which adds a leaf to every leaf in the caterpillar?

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