Spring 4-29-2022

Einstein's Equations, Lagrangians for General Relativity, and ADM Formalism

Timothy Corbett

Einstein's Equations, Lagrangians for General Relativity, and ADM Formalism

Timothy K. Corbett

submitted in partial fulfillment of the requirements for Honors in Mathematics at the University of Mary Washington

Fredericksburg, Virginia

April 2022

This thesis by **Timothy K. Corbett** is accepted in its present form as satisfying the thesis requirement for Honors in Mathematics.

DATE	APPROVED
4/28/2022	Dr. Yuan-Jen Chiang, Professor of Mathematics, thesis advisor
	Dr. Julius Esunge, Professor and Chair of Mathematics, committee member
	Dr. Suzanne Sumner, Professor of Mathematics, committee member

Contents

1 Preliminaries **1**
1.1 Basic Concepts of Manifolds 1
1.2 Riemann Curvature Tensor 2
1.3 Differential Forms 3

2 Einstein's Equation **4**
2.1 The Stress-Energy Tensor 4
2.2 The Einstein Tensor 4
2.3 The Schwarzschild Metric 5

3 Cosmology **8**
3.1 Robertson-Walker Cosmological Model 8
3.2 Cosmological Evolution Equations 9

4 Lagrangians for General Relativity **11**

5 ADM Formalism **14**
5.1 Extrinsic Curvature 14
5.2 The Gauss-Codazzi Equations 16
5.3 Canonical Quantization 19

Abstract

Einstein's equations describe the relation of spacetime curvature and present matter. We will consider the case of a Lorentzian manifold diffeomorphic to $\mathbb{R} \times \Sigma$, where time $t \in \mathbb{R}$ and the three-dimensional manifold Σ represents space. In a homogeneous and isotropic universe, the three-dimensional manifold Σ of the Lorentzian manifold has Riemann curvature tensor ${}^{(3)}R = KI$, where K is a constant and I is the 3×3 identity matrix. Space is flat when $K = 0$, spherical when K is positive, and hyperbolic when K is negative. In this paper, we will show models agreeing with each. We will derive the Schwarzschild metric, find Einstein's equations from the Lagrangian, and analyze them using Arnowitt-Deser-Misner formalism.

1 Preliminaries

The subjects covered in this paper are important features in general relativity. While in special relativity, we can approximate the geometry of spacetime to be $\mathbb{R}^4$, this assumption is deemed inappropriate in general relativity, as, when making this assumption, there are errors in predicting the gravitation between two particles. We use a Lorentzian metric, which forces that spacetime be curved in the presence of a gravitational field. "The world lines of freely falling bodies in a gravitational field are simply the geodesics of the spacetime metric."[10] Because of this, we cannot meaningfully describe gravitation as a force field, but, rather, we find meaning in the relative gravitational forces between free falling bodies. Now, what is a "geodesic"? To answer this question, we must cover some basic differential geometry for the sake of the reader.

1.1 Basic Concepts of Manifolds

Imagine we have a surface M, a two-dimensional manifold. The surface can be any shape of membrane - a sphere, a torus, a cylinder, a paraboloid, a hyperboloid, a flag flapping in the wind, etc. For a given point p on our surface, there is an open neighborhood $\mathcal{U}_p$ of p. The surface, as a whole, is composed of a collection of these open neighborhoods $\{\mathcal{U}_p\}_{p\in M}$. A coordinate patch is a C^k one-to-one function $\mathbf{x}$: $\mathcal{U}_p \longrightarrow \mathbb{R}^3$ for some $k \geq 1$, for which the coordinates of $\mathcal{U}_p$ are u^1, u^2 and

$$\frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} = \mathbf{x}_1 \times \mathbf{x}_2 \neq 0. \tag{1.1}$$

The unit normal at a $p = \mathbf{x}(a, b)$ is equal to

$$n(a, b) = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}. \tag{1.2}$$

This normal is perpendicular to the tangent plane T_pM at p. The tangent plane sitting at $p = \mathbf{x}(a, b)$ is the plane perpendicular to $\mathbf{x}_1(a, b) \times \mathbf{x}_2(a, b)$. Therefore, if $\mathbf{x}(a, b) = (x^1, y^1, z^1)$ and $\mathbf{x}_1 \times \mathbf{x}_2 = (x^2, y^2, z^2)$, then the tangent plane would be

$$x^2(x - x^1) + y^2(y - y^1) + z^2(z - z^1) = 0. \tag{1.3}$$

We denote the tangent space at p, the set of all tangent vectors of a surface passing through p, as T_pM. The metric is defined as the map,

$$g : T_pM \times T_pM \longrightarrow \mathbb{R}, \tag{1.4}$$

for which its components, the first fundamental form of a surface, are

$$g_{ij}(u^1,u^2) = \langle \mathbf{x}_i(u^1,u^2)\cdot \mathbf{x}_j(u^1,u^2)\rangle, i=1,2, j=1,2. \tag{1.5}$$

In the Lorentzian manifold $M=\mathbb{R}\times\Sigma$, the metric takes the following form:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & {}^3g_{ij} & \\ 0 & & & \end{pmatrix}, i,j=1,2,3. \tag{1.6}$$

where ${}^{(3)}g_{ij}$ is the metric on the three-dimensional manifold Σ. When we use an inverse of a matrix, typically we will either raise or lower the indices. In the case of the metric, we define the reverse in our notation as $g^{\mu\nu}$. The second fundamental form is defined for tangent vectors $\mathbf{X}=\Sigma X^i\mathbf{x}_i$,$\mathbf{Y}=\Sigma Y^i\mathbf{y}_i$

$$\mathbf{II}(\mathbf{X},\mathbf{Y}) = \Sigma_{i,j}L_{ij}X^iY^j, \tag{1.7}$$

where coefficients $L_{ij}=\langle\mathbf{x}_{ij},n\rangle$. The Christoffel symbols, the metric connections, are intrinsic functions on $\mathcal{U}_p$ defined as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\Sigma_\sigma g^{\rho\sigma}\{\frac{\partial g_{\nu\sigma}}{\partial x^\mu}+\frac{\partial g_{\mu\sigma}}{\partial x^\mu}-\frac{\partial g_{\nu\sigma}}{\partial x^\mu}\}. \tag{1.8}$$

Let $\mathbf{x}$: $\mathcal{U}_p\longrightarrow\mathbb{R}^3$ be a local surface. Then we have Gauss's formulas:

$$\mathbf{x}_{ij} = L_{ij}\mathbf{n}+\Sigma_k\Gamma^k_{ij}\mathbf{x}_k. \tag{1.9}$$

A geodesic on a surface is a unit speed curve with geodesic curvature equal to zero everywhere. The geodesic curvature of a unit speed curve is the component of the second derivative of the curve in the direction of $\mathbf{S}=n\times T$ for tangent vector T. It is represented by $K_g=[n,T,T']$. For any unit speed curve, $\gamma(s)=\mathbf{x}(\gamma^1(s),\gamma^2(s))$, the normal curvature is

$$\kappa_n = \Sigma_{i,j}L_{ij}(\gamma^i)'(\gamma^j)' \tag{1.10}$$

and the geodesic curvature is

$$K_g\mathbf{S} = \Sigma_k[(\gamma^k)''+\Sigma_{i,j}\Gamma^k_{ij}(\gamma^i)'(\gamma^j)']\mathbf{x}_k. \tag{1.11}$$

1.2 Riemann Curvature Tensor

For a point on a vector space V, we can define a vector bundle above the vector space. For this vector bundle, we can define a connection, denoted by D. In particular, if this V is a tangent space, the Levi-Cevita connection, a unique, metric-preserving and torsion free connection on T_pM for metric g, is given the symbol ∇. If $\nabla g_{\mu\nu}=0$, ∇ is a metric connection for $g_{\mu\nu}$ on M. Below is a sample of the Lie algebra of this connection with vector fields ∂_μ:

$$\nabla_\alpha\partial_\beta-\nabla_\beta\partial_\alpha = [\partial_\beta,\partial_\alpha]. \tag{1.12}$$

We also witness some important algebra between ∇ and something we call the Riemann curvature tensor, the curvature of ∇:

$$R(u,v)w = (\nabla_u\nabla_v-\nabla_v\nabla_u-\nabla_{[u,v]})w, \tag{1.13}$$

$$R(\partial_\beta, \partial_\gamma)\partial_\delta = (\nabla_\beta \nabla_\gamma - \nabla_\gamma \nabla_\beta)\partial_\delta. \tag{1.14}$$

We can define the Riemann tensor itself in terms of Christoffel symbols,

$$R^\alpha_{\beta\gamma\delta} = \partial_\beta \Gamma^\alpha_{\gamma\delta} - \partial_\gamma \Gamma^\alpha_{\beta\delta} + \Gamma^\sigma_{\gamma\delta}\Gamma^\alpha_{\beta\sigma} - \Gamma^\sigma_{\beta\delta}\Gamma^\alpha_{\gamma\sigma}. \tag{1.15}$$

When we contract the indices of the Riemann tensor, we can get the Ricci tensor, $R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta}$, where we can describe this as

$$R_{\mu\nu} = \Sigma_\lambda \frac{\partial}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial}{\partial x^\mu}(\Sigma_\lambda \Gamma^\lambda_{\lambda\nu}) + \Sigma_{\alpha,\lambda}(\Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\alpha\lambda} - \Gamma^\alpha_{\lambda\nu}\Gamma^\lambda_{\alpha\mu}). \tag{1.16}$$

We then can find the scalar curvature, $R = R^\alpha_\alpha$, which is important when finding the action later. We can take a moment to mention a few symmetries of the Riemann tensor:

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda} R^\lambda_{\beta\gamma\delta}, \tag{1.17}$$
$$R^\lambda_{\beta\gamma\delta} = -R^\lambda_{\gamma\beta\delta}, \tag{1.18}$$
$$R_{\alpha\beta\gamma\delta} = -R_{\delta\beta\gamma\alpha}, \tag{1.19}$$
$$R^\lambda_{[\beta\gamma\delta]} = 0. \tag{1.20}$$

1.3 Differential Forms

One other small detail we must go over is the differential form. A 1-form is an exterior derivative of a function f. We can define derivative this as

$$df = \partial_\mu f dx^\mu. \tag{1.21}$$

Since

$$dx^\mu(\partial_\nu) = \partial_\nu x^\mu = \delta^\mu_\nu, \tag{1.22}$$

we can define a 1-form ω on $\mathbb{R}^n$ when

$$\omega_\mu = \omega(\partial_\mu), \tag{1.23}$$

as

$$\omega = \omega_\mu dx^\mu. \tag{1.24}$$

We must mention the star operator, as it is used briefly later. When we take the cross product of two vectors, in actuality, we are taking the wedge product and applying the star operator. The wedge product will look similar to the cross product:

$$\omega \wedge \mu = -\mu \wedge \omega. \tag{1.25}$$

The following is a list of star operators in the dx, dy, dz basis:

$$\star : dx \wedge dy \mapsto dz \tag{1.26}$$
$$\star : dy \wedge dz \mapsto dx \tag{1.27}$$
$$\star : dz \wedge dx \mapsto dy. \tag{1.28}$$

We can rewrite Maxwell's equations for electromagnetism in these terms as

$$\star d \star F = J, \tag{1.29}$$
$$d \star J = 0, \tag{1.30}$$

when J is the current density and

$$F = B + E \wedge dt, \tag{1.31}$$

for magnetic and electric fields B and E. These equations are important because much of the mathematics from electromagnetism can be used to justify certain relationships in general relativity, as we will see in the section *Einstein's Equation*.

Furthermore, we may find the Bianchi identity for a field F and connection D useful,

$$d_D F = 0, \tag{1.32}$$

when d_D is the exterior covariant derivative.

2 Einstein's Equation

2.1 The Stress-Energy Tensor

Einstein's equation describes how spacetime is curved by the presence of matter (anything which possesses energy or momentum). This equation is a (0,2) tensor of chosen local coordinates $t = x^0$, x^1, x^2, x^3. It is a symmetric tensor, so entries of the stress-energy tensor $T_{\mu\nu} = T_{\nu\mu}$. We use the stress-energy tensor to describe the law of conservation of energy-momentum on curved spacetime as

$$g^{\mu\lambda}\nabla_\lambda T_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0. \tag{2.1}$$

We know from electromagnetism that $d \star J = 0$ for the current 1-form J, and we can show that $\star d \star J = -\nabla^\mu J_\mu$, which can also be written $\nabla^\mu J_\mu = 0$. We are working in Minkowski spacetime, meaning the Levi-Civita connection is the flat connection, $\partial^\mu J_\mu = 0$. The following is our continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0. \tag{2.2}$$

This equation says any divergence in J gives a corresponding change in charge density ρ. Similarly, $T\mu\nu$ is divergence free on Minkowski spacetime, and with $\nabla^\mu T_{\mu\nu} = 0$, if $\nu = 0$, we get the law of conservation of energy:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{i0}}{\partial x^i} = 0 \tag{2.3}$$

and the law of conservation of momentum when j=1,2,3:

$$\frac{\partial T^{0j}}{\partial t} + \frac{\partial T^{ij}}{\partial x^i} = 0. \tag{2.4}$$

2.2 The Einstein Tensor

We want equations for gravity which are consistent with the law of conservation of energy, so we choose for the stress energy tensor to be set equal to some divergence-free tensor depending only on the curvature of spacetime, $C_{\mu\nu} = T_{\mu\nu}$. The Riemann tensor is such. We can find the simplest divergence-free symmetric result using the Bianchi identity. To get there, we start with the Jacobi identity for any vector fields α, β, γ on spacetime:

$$[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + [\nabla_\beta, [\nabla_\gamma, \nabla_\alpha]] + [\nabla_\gamma, [\nabla_\alpha, \nabla_\beta]] = 0. \tag{2.5}$$

The following Lie algebra is true: $[\nabla_\mu, \nabla_\nu] = R(\partial_\mu, \partial_\nu)$. This relationship allows the following substitution:

$$[\nabla_\alpha, R(\partial_\beta, \partial_\gamma)] + [\nabla_\beta, R(\partial_\gamma, \partial_\alpha)] + [\nabla_\gamma, R(\partial_\alpha, \partial_\beta)] = 0. \tag{2.6}$$

We change this equation in terms of curvature as the Riemann tensor. We describe $R^\alpha_{\beta\gamma\delta}$ with $R^\alpha_{\beta\gamma\delta} = -R^\alpha_{\gamma\beta\delta}$ as $\mathcal{R}$ to get a nice form of the Bianchi identity, d_∇ being the exterior covariant derivative from the Levi-Civita connection:

$$d_\nabla \mathcal{R} = 0. \tag{2.7}$$

Another way we can describe this equation is in local coordinates. First, the exterior derivative of a p-form is $d(\omega_I dx^I) = (\partial_\mu \omega_I) dx^\mu \wedge dx^I = (\nabla_\mu \omega_I) dx^\mu \wedge dx^I$. This substitution changes the Bianchi identity to

$$\nabla_{[\alpha} R^\lambda_{\beta\gamma]\delta} = 0. \tag{2.8}$$

By contracting indices, we have $\nabla_{[\alpha} R^\alpha_{\beta\gamma]\delta} = 0$. Using the explicit antisymmetrization, contraction and raising of indices, we can perform the following,

$$\begin{aligned}
0 &= \nabla_\alpha R^\alpha_{\beta\gamma\delta} + \nabla_\beta R^\alpha_{\gamma\alpha\delta} + \nabla_\gamma R^\alpha_{\alpha\beta\delta} && (2.9)\\
&= \nabla_\alpha R^\alpha_{\beta\gamma\delta} + \nabla_\beta R^\alpha_{\gamma\alpha\delta} - \nabla_\gamma R^\alpha_{\beta\alpha\delta} && (2.10)\\
&= \nabla^\alpha R_{\alpha\beta\gamma\delta} + \nabla_\beta R_{\gamma\delta} - \nabla_\gamma R_{\beta\delta} && (2.11)\\
&= \nabla^\alpha R_{\delta\gamma\beta\alpha} + \nabla_\beta R_{\gamma\delta} - \nabla_\gamma R_{\beta\delta} && (2.12)\\
&= \nabla^\alpha R_{\gamma\alpha} + \nabla^\beta R_{\gamma\beta} - \nabla_\gamma R && (2.13)\\
&= \nabla^\alpha R_{\gamma\alpha} + \nabla^\alpha R_{\gamma\alpha} - \nabla_\gamma R && (2.14)\\
&= 2\nabla^\alpha R_{\gamma\alpha} - \nabla_\gamma R && (2.15)\\
&= \nabla^\alpha R_{\gamma\alpha} - \frac{1}{2}\nabla_\gamma R && (2.16)\\
&= \nabla^\alpha (R_{\gamma\alpha} - \frac{1}{2} g_{\gamma\alpha} R). && (2.17)
\end{aligned}$$

The last is because $\nabla^\alpha g_{\gamma\alpha} = 0$. We then have $G_{\mu\nu} = R_{\gamma\alpha} - \frac{1}{2} g_{\gamma\alpha} R$, the Einstein tensor. Now, we can say

$$\nabla^\mu G_{\mu\nu} = 0. \tag{2.18}$$

We find that the Einstein tensor is the result we were seeking. This result allows the Einstein tensor to be a multiple of the stress-energy tensor.

$$G_{\mu\nu} = 8\pi\kappa T_{\mu\nu}. \tag{2.19}$$

Now, since $\nabla^\mu g_{\mu\nu} = 0$, we could add a term proportional to the metric such that

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\kappa T_{\mu\nu}, \tag{2.20}$$

Λ being the cosmological constant.

2.3 The Schwarzschild Metric

With the Schwarzschild Solution, Einstein's equation will describe the exterior gravitational field (spacetime geometry of the surrounding empty space) of a static, spherically symmetric body. In this case, spacetime coordinates will be t, r, θ, and ϕ. The metric of a static, spherically symmetric spacetime is of this form:

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.21}$$

An orthonormal basis for this metric is shown here,

$$(e_0)_\mu = f^{1/2}(dt)_\mu \tag{2.22}$$
$$(e_1)_\mu = h^{1/2}(dr)_\mu \tag{2.23}$$
$$(e_2)_\mu = r(d\theta)_\mu \tag{2.24}$$
$$(e_3)_\mu = rsin\theta(d\phi)_\mu, \tag{2.25}$$

then we use the derivative ∂_μ where $f' = df/dr$,

$$\partial_{[\mu}(e_0)_{\nu]} = \frac{1}{2}f^{-1/2}f'(dr)_{[\mu}(dt)_{\nu]} \tag{2.26}$$
$$\partial_{[\mu}(e_1)_{\nu]} = 0 \tag{2.27}$$
$$\partial_{[\mu}(e_2)_{\nu]} = (dr)_{[\mu}(d\theta)_{\nu]} \tag{2.28}$$
$$\partial_{[\mu}(e_3)_{\nu]} = sin\theta(dr)_{[\mu}(d\phi)_{\nu]}rcos\theta(d\theta)_{[\mu}(d\phi)_{\nu]}. \tag{2.29}$$

Now, using

$$\partial_{[a}(e_\sigma)_{b]} = \Sigma_{\mu,\nu}\eta^{\mu\nu}(e_\mu)_{[a}\omega_{b]\sigma\nu} \tag{2.30}$$

we solve for the connection 1-forms $\omega_{\mu\alpha\beta} = -\omega_{\mu\beta\alpha}$:

$$\frac{1}{2}f^{-1/2}f'(dr)_{[\mu}(dr)_{\nu]} = h^{1/2}(dr)_{[\mu}\omega_{\nu]01} + r(d\theta)_{[\mu}\omega_{\nu]02} + rsin\theta(d\phi)_{[\mu}\omega_{\nu]03} \tag{2.31}$$
$$0 = f^{1/2}(dt)_{[\mu}\omega_{\nu]01} + r(d\theta)_{[\mu}\omega_{\nu]12} + rsin\theta(d\phi)_{[\mu}\omega_{\nu]13} \tag{2.32}$$
$$(dr)_{[\mu}(d\theta)_{\nu]} = -f^{1/2}(dt)_{[\mu}\omega_{\nu]20} + h^{1/2}(dr)_{[\mu}\omega_{\nu]21} + rsin\theta(d\phi)_{[\mu}\omega_{\nu]23} \tag{2.33}$$
$$sin\theta(dr)_{[\mu}(d\phi)_{\nu]} + rcos\theta(d\theta)_{[\mu}(d\phi)_{\nu]} = -f^{1/2}(dt)_{[\mu}\omega_{\nu]30} + h^{1/2}(dr)_{[\mu}\omega_{\nu]31}r(d\theta)_{[\mu}\omega_{\nu]32}. \tag{2.34}$$

We offer the guess $\omega_{\nu 02} = \omega_{\nu 03} = 0$, so

$$\omega_{\nu 01} = \frac{1}{2}\frac{f'}{(fh)^{1/2}}(dt)_\nu + \alpha_1(dr)_\nu. \tag{2.35}$$

in which α_1 is an undetermined function.

We plug these terms into equation (2.32) from just above to find that $\alpha_1 = 0$. We can also guess from (2.32) that

$$\omega_{\nu 12} = \alpha_2(d\theta)_\nu + \alpha_3(d\phi)_\nu, \tag{2.36}$$
$$\omega_{\nu 13} = \alpha_4(d\phi)_\nu + \frac{\alpha_3}{sin\theta}(d\theta)_\nu. \tag{2.37}$$

We plug everything into (2.33) to get

$$\alpha_2 = -h^{-1/2}, \tag{2.38}$$
$$\omega_{\nu 23} = -\frac{h^{1/2}}{rsin\theta}\alpha_3(dr)_\nu + \alpha_5(d\phi)_\nu. \tag{2.39}$$

Next we substitute into (2.34) and have

$$\alpha_3 = 0, \tag{2.40}$$
$$\alpha_4 = -h^{-1/2} sin\theta, \tag{2.41}$$
$$\alpha_5 = -cos\theta. \tag{2.42}$$

There have been no bumps in the road, so we have all of our values:

$$\omega_{\nu 01} = \frac{f'}{2(fh)^{1/2}}(dt)_\nu, \tag{2.43}$$
$$\omega_{\nu 12} = -h^{-1/2}(d\theta)_\nu, \tag{2.44}$$
$$\omega_{\nu 13} = -h^{-1/2} sin\theta (d\theta)_\nu, \tag{2.45}$$
$$\omega_{\nu 23} = -cos\theta (d\phi)_\nu. \tag{2.46}$$

Now, we can find all of the Riemann tensors:

$$R_{\mu\nu 01} = -R_{\mu\nu}10 = \frac{d}{dr}[(fh)^{-1/2} f'](dr)_{[\mu}(dt)_{\nu]}, \tag{2.47}$$
$$R_{\mu\nu 02} = f^{-1/2} h^{-1} f'(d\theta)_{[\mu}(dt)_{\nu]}, \tag{2.48}$$
$$R_{\mu\nu 03} = f^{-1/2} h^{-1} f' sin\theta (d\phi)_{[\mu}(dt)_{\nu]}, \tag{2.49}$$
$$R_{\mu\nu 12} = h^{-3/2} h'(dr)_{[\mu}(d\theta)_{\nu]}, \tag{2.50}$$
$$R_{\mu\nu 13} = sin\theta h^{-3/2} h'(dr)_{[\mu}(d\phi)_{\nu]}, \tag{2.51}$$
$$R_{\mu\nu 23} = 2(1-h^{-1}) sin\theta (d\theta)_{[\mu}(d\phi)_{\nu]}. \tag{2.52}$$

We find the Ricci tensors from the Riemann tensors and set them equal to zero for the vacuum Einstein equation when $R_{\alpha\beta} = R_{\mu\nu}(e_\alpha)^\mu (e_\beta)^\nu$:

$$0 = R_{00} = R^1_{010} + R^2_{020} + R^3_{030} = \frac{1}{2}(fh)^{-1/2}\frac{d}{dr}[(fh)^{-1/2} f'] + (rfh)^{-1} f', \tag{2.53}$$
$$0 = R_{11} = -\frac{1}{2}(fh)^{-1/2}\frac{d}{dr}[(fh)^{-1/2} f'] + (rh^2)^{-1} h, \tag{2.54}$$
$$0 = R_{22} = -\frac{1}{2}(rfh)^{-1} f' + \frac{1}{2}(rh^2)^{-1} h' + r^{-2}(1-h^{-1}) = R_{33}. \tag{2.55}$$

The off-diagonals of $R_{\alpha\beta}$ vanish as well.

We see from the sum

$$R_{00} + R_{11} = \frac{f'}{f} + \frac{h'}{h} = 0, \tag{2.56}$$

that $f = Kh^{-1} \Rightarrow t \longrightarrow \sqrt{K}t$, we set constant $K = 1$. From these parameters,

$$R_{22} = -f' + \frac{1-f}{r} = 0, \tag{2.57}$$

or

$$\frac{d}{dr}(rf) = 1. \tag{2.58}$$

Now, $f = 1 + \frac{C}{r}$ when C is constant.
We can now write out our general solution,

$$ds^2 = -(1 + \frac{C}{r})dt^2 + (1 + \frac{C}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \quad (2.59)$$

When $r \longrightarrow \infty$, this solution reaches flatness asymptotically, so we can say that the behavior of a body agrees with Newtonian behavior when r is large, or $C = -\frac{2GM}{c^2}$. Therefore, this result is our Schwarzschild metric,

$$ds^2 = -(1 - \frac{2GM}{rc^2})dt^2 + (1 - \frac{2GM}{rc^2})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \quad (2.60)$$

Here, r is our critical radius in which, if the entirety of the mass of the spherical object were to be scrunched, then not even light could escape its gravity.[6] The equation is essentially setting light speed to be the escape velocity from the gravitational field of a mass. The surface at this radius is called the horizon and appears to be a black, circular void to the observer.

3 Cosmology

3.1 Robertson-Walker Cosmological Model

It has been a long-time assumption that our universe is homogeneous and isotropic. If we were situated anywhere else in the universe, the laws of physics that we observe here would apply there, and, regardless of where we look, observations which yield results should not change. "There are no preferred directions in space."[10] Through the ages of technology broadening our horizons, we have continued to see this assumption holds. We can define homogeneity and isotropy mathematically. Let us say there is a one-parameter family of spacelike hypersurfaces we call Σ_t. We will take a look at this figure from General Relativity by R. Wald, (Fig. C1). Spacetime is spatially homogeneous if for all t, and for any p,q $\in \Sigma_t$, there is an isometry of the spacetime metric (some ϕ s.t. $(\phi * g)_{ab} = g_{ab}$) which takes p into q.[10] Next, we look to another figure from General Relativity, (Fig. C2). A spacetime is isotropic at each point if there exist timelike curves with tangents u^a orthogonal to Σ_t, such that any point p connecting tangent vectors to Σ_t, s_1^a, $s_2^a \in V_p$, there is an isometry of g_{ab} for which p and u^a are fixed, but s_1^a rotates into s_2^a. We see, therefore, that we cannot construct a preferred tangent vector orthogonal to u^a.

Because of the restrictions of homogeneity and isotropy, the Riemann tensor can of Σ_t be described as a multiple of the identity operator,

$$^{(3)}R_{\alpha\beta}^{\gamma\delta} = \kappa I = \delta_{[\alpha}^{\gamma}\delta_{\beta]}^{\delta}. \quad (3.1)$$

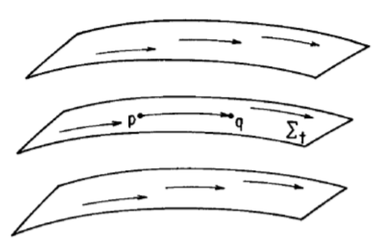

Fig. C1

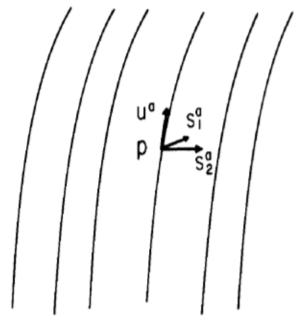

Fig. C2

We then lower the indices,

$$R_{\alpha\beta\gamma\delta} = \kappa h_{\gamma[\alpha} h_{\beta]\delta}. \tag{3.2}$$

Since κ is constant, which is implied by homogeneity, a space where this property is satisfied is called a space of constant curvature. Any two spaces of constant curvature of equal dimension, metric, and κ are locally isometric.

The 3-sphere spacial geometry provides all positive κ, negative κ comes from three-dimensional hyperboloids, and $\kappa = 0$ for flat space. Given that the isotropic and homogeneous surfaces are orthogonal to each other, we can define the four-dimensional metric for spacetime as

$$g_{\mu\nu} = -u_\mu u_\nu + h_{\mu\nu}(t), \tag{3.3}$$

for which $h_{\mu\nu}$ is the metric of a spatial geometry from the selection below for a proper time τ carried by an isotropic observer:

$$ds^2 = -d\tau^2 + a^2(\tau)(d\psi^2 + sin^2\psi(d\theta^2 + sin^2\theta d\phi^2)), \tag{3.4}$$

$$ds^2 = -d\tau^2 + a^2(\tau)(dx^2 + dy^2 + dz^2), \tag{3.5}$$

$$ds^2 = -d\tau^2 + a^2(\tau)(d\psi^2 + sinh^2\psi(d\theta^2 + sin^2\theta d\phi^2)), \tag{3.6}$$

for which (3.4) is the spherical metric, (3.5) is the flat space metric, and (3.6) is the hyperbolic space metric.

These metrics are called the Robertson-Walker cosmological model. We want to use the spacetime metric to find ordinary differential equations to predict the general evolution for homogeneous, isotropic cosmology.

3.2 Cosmological Evolution Equations

First, we define the stress-energy tensor of matter in the present universe to be the average mass density of matter times these tangent vectors for points on observation lines, perpendicular to foliating hypersurfaces of spacetime, $T_{\mu\nu} = \rho u_\mu u_\nu$. We must also include a term involving thermal radiation pressure P in our expression, so we multiply it by the metric of the hypersurfaces:

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu). \tag{3.7}$$

Due to our two restrictions of homogeneity and isotropy, the only components we must compute of the Einstein tensor are $G_{\tau\tau}$ and G_{ss}, so these terms are our only components for Einstein's equations:

$$G_{\mu\nu}u^{\mu}u^{\nu} = G_{\tau\tau} = 8\pi T_{\tau\tau} = 8\pi\rho, \tag{3.8}$$
$$G_{\mu\nu}s^{\mu}s^{\nu} = G_{ss} = 8\pi T_{ss} = 8\pi\rho. \tag{3.9}$$

First, we compute these expressions for flat spatial geometry, $ds^2 = -d\tau^2 + a^2(\tau)(dx^2+dy^2+dz^2)$. The Christoffel components are, $\dot{a}$ being with respect to τ,

$$\Gamma^{\tau}_{xx} = \Gamma^{\tau}_{yy} = \Gamma^{\tau}_{zz} = a\dot{a}, \tag{3.10}$$
$$\Gamma^{x}_{x\tau} = \Gamma^{x}_{\tau x} = \Gamma^{y}_{y\tau} = \Gamma^{y}_{\tau y} = \Gamma^{z}_{z\tau} = \Gamma^{z}_{\tau z} = \frac{\dot{a}}{a}. \tag{3.11}$$

From the definition of the Ricci tensor, we have

$$R_{\tau\tau} = -\frac{3\ddot{a}}{a}, \tag{3.12}$$
$$R_{ss} = a^{-2}R_{xx} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}. \tag{3.13}$$

Therefore, $\frac{1}{2}R = \frac{1}{2}(-R_{\tau\tau} + 3R_{ss}) = 3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})$.
We then compute

$$G_{\tau\tau} = -3\frac{\ddot{a}}{a} + 3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) = 3\frac{\dot{a}^2}{a^2} = 8\pi\rho, \tag{3.14}$$
$$G_{ss} = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - 3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = 8\pi P. \tag{3.15}$$

We rewrite G_{ss} to be $3\frac{\ddot{a}^2}{a^2} = -4\pi(\rho + 3P)$.
Next, we compute $G_{\tau\tau}$ and G_{ss} for spherical and hyperbolic space, k being either ± 1.

$$\Gamma^{\tau}_{\theta\theta} = r^2 a\dot{a} \tag{3.16}$$

$$\Gamma^{r}_{\tau r} = \Gamma^{r}_{r\tau} = \Gamma^{\theta}_{\tau\theta} = \Gamma^{\theta}_{\theta\tau} = \Gamma^{\phi}_{\tau\phi} = \Gamma^{\phi}_{\phi\tau} = \frac{\dot{a}}{a} \tag{3.17}$$

$$\Gamma^{r}_{\theta\theta} = -r(1 - kr^2) \tag{3.18}$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \Gamma^{\phi}_{r\phi} = \Gamma^{\theta}_{\phi r} = \frac{1}{r} \tag{3.19}$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{cos\theta}{sin\theta} \tag{3.20}$$

$$R_{\tau\tau} = -\frac{3\dot{a}'}{a} \tag{3.21}$$

$$R_{rr} = \frac{2k + 2\dot{a}^2 + \dot{a}\ddot{a}}{1 - kr^2} \tag{3.22}$$

$$R_{\theta\theta} = -1 + 2kr^2 - cot^2\theta + csc^2\theta + 2r^2\dot{a}^2 + r^2 a\ddot{a} \tag{3.23}$$

$$R_{\phi\phi} = r^2 sin^2\theta(2k + 2\dot{a}^2 + a\ddot{a}) \tag{3.24}$$

$$G_{\tau\tau} = R_{\tau\tau} + \frac{1}{2}R = \frac{3k}{a^2} + \frac{3\dot{a}^2}{a^2} = 8\pi\rho \tag{3.25}$$

$$R_{ss} = \frac{R_{\mu\nu}X^\mu X^\nu}{g_{\mu\nu}X^\mu X^\nu} = \frac{R_\phi}{g_{\phi\phi}} \tag{3.26}$$

$$G_{ss} = -\frac{k}{a^2} - \frac{\dot{a}^2}{a} - \frac{2\ddot{a}}{a} = 8\pi P \tag{3.27}$$

The general evolution differential equations are found to be

$$3\frac{\dot{a}^2}{a^2} = 8\pi\rho - 3\frac{k}{a^2}, \tag{3.28}$$

$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3P). \tag{3.29}$$

We see that the universe must always be either expanding or contracting.

We will not work out the exact solutions for our three geometries in this paper, but, for the curious reader, the below table displays exact solutions taken from General Relativity by R. Wald, for which $\eta = \int d\eta = \int \frac{dt}{a}$:

Dust and Radiation Filled Robertson-Walker Cosmologies		
Spatial Geometry	"Dust"; $P = 0$	Radiation; $P = \frac{1}{3}\rho$
3-sphere, $k = +1$	$a = \frac{4\pi\rho a^4}{3}(1 - \cos\eta)$ $\tau = \frac{4\pi\rho a^4}{3}(\eta - \sin\eta)$	$a = \sqrt{\frac{8\pi\rho a^4}{3}}[1 - (1 - \tau/\sqrt{\frac{8\pi\rho a^4}{3}})^2]^{1/2}$
Flat, $k = 0$	$a = (6\pi\rho a^3)^{1/3}\ \tau^{2/3}$	$a = (\frac{32\pi\rho a^4}{3})^{1/4}\ \tau^{1/2}$
Hyperboloid, $k = -1$	$a = \frac{4\pi\rho a^4}{3}(\cosh\eta - 1)$ $\tau = \frac{4\pi\rho a^4}{3}(\sinh\eta - \eta)$	$a = \sqrt{\frac{8\pi\rho a^4}{3}}[(1 + \tau/\sqrt{\frac{8\pi\rho a^4}{3}})^2 - 1\,]^{1/2}$

.

4 Lagrangians for General Relativity

We want to derive Einstein's equation from an action principle for gravity. The action principle is a path along which an action is minimized, the action being the integral of the Lagrangian, the kinetic energy minus the potential energy of a particle as a function of time.

When spacetime is an oriented manifold M on which there is a metric g, the Lagrangian for general relativity is Rvol, R being the Ricci scalar curvature of g and vol g's volume form. The Einstein-Hilbert action is then

$$S(g) = \int_M R\text{vol}. \tag{4.1}$$

This integral can be rewritten in local coordinates as $S(g) = \int_M R\sqrt{|\det g|}d^n x = \int_M R\sqrt{-\det g}d^n x$. We are working in the Lorentzian case, so we calculate the variation of the action as

$$\delta S(g) = \frac{d}{ds} S(g + s\delta g)|_{s=0}, \tag{4.2}$$

with Lorentzian matrix g, (0,2)-tensor δg which vanishes outside a compact set when M is such, and s is a real number that will form a path of metrics through g when in this expression as it varies. This method is how we find the variation of any quantity depending on g, so we start with our original definition

$$\delta S = \int_M \delta(R\text{vol}) = \int_M (\delta R)\text{vol} + R\delta\text{vol}, \tag{4.3}$$

and we translate it into local coordinates:

$$\delta S = \int_M \delta(R\sqrt{|\det g|}d^n x) = \int_M (\delta R)\sqrt{|\det g|} + R\delta\sqrt{|\det g|}d^n x. \tag{4.4}$$

We utilize the fact that, for any two matrices A and B,

$$\begin{aligned} \frac{d}{ds}\det(A + sB)|_{s=0} &= \frac{d}{ds}\det(A)\det(1 + sA^{-1}B)|_{s=0} \\ &= \det(A) tr(A^{-1}B), \end{aligned} \tag{4.5}$$

to say

$$\begin{aligned} \delta \det g &= \frac{d}{ds}\det(g + s\delta g)|_{s=0} \\ &= \det g g^{-1}\delta g \\ &= (\det g) g^{\alpha\beta}\delta g_{\alpha\beta}. \end{aligned} \tag{4.6}$$

Since trace $g^{\alpha\beta}g_{\alpha\beta}$ is equal to its number of dimensions, $\delta(g^{\alpha\beta}g_{\alpha\beta}) = 0$, so $g^{\alpha\beta}\delta g_{\alpha\beta} = -g_{\alpha\beta}\delta g^{\alpha\beta}$. Therefore we can perform the following,

$$\delta(\det g) = -(\det g)g_{\alpha\beta}\delta g^{\alpha\beta} \Rightarrow \delta\sqrt{-\det g} = -\frac{1}{2}\sqrt{-\det g}g_{\alpha\beta}\delta g^{\alpha\beta}. \tag{4.7}$$

This expression is just δvol$= -\frac{1}{2}g_{\alpha\beta}(\delta g^{\alpha\beta})$vol.

We begin calculation of the Ricci scalar, δR,

$$\delta R = \delta R^\alpha_\alpha = \delta(g^{\alpha\beta}R_{\alpha\beta}) = (\delta g^{\alpha\beta})R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}. \tag{4.8}$$

We need to find $\delta R_{\alpha\beta} = \delta R^\gamma_{\alpha\gamma\beta}$. This calculation means first finding the variation of the Christoffel symbols:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\eta}(\partial_\beta g_{\gamma\eta} + \partial_\gamma g_{\beta\eta} - \partial_\eta g_{\beta\gamma}), \tag{4.9}$$

the variation being

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\eta}(\nabla_\beta\delta g_{\gamma\eta} + \nabla_\gamma\delta g_{\beta\eta} - \nabla_\eta\delta g_{\beta\gamma}). \tag{4.10}$$

We can now look for the variation of the Riemann tensor,

$$R^{\alpha}_{\beta\gamma\eta} = \partial_\beta \Gamma^{\alpha}_{\gamma\eta} - \partial_\gamma \Gamma^{\alpha}_{\beta\eta} + \Gamma^{\sigma}_{\gamma\eta}\Gamma^{\alpha}_{\beta\sigma} - \Gamma^{\sigma}_{\beta\eta}\Gamma^{\alpha}_{\gamma\sigma}, \tag{4.11}$$

as

$$\delta R^{\alpha}_{\beta\gamma\eta} = \nabla_\beta \delta\Gamma^{\alpha}_{\gamma\eta} - \nabla_\gamma \delta\Gamma^{\alpha}_{\beta\eta}. \tag{4.12}$$

or

$$\delta R_{\alpha\beta} = \delta R^{\gamma}_{\alpha\gamma\beta} = \nabla_\alpha \delta\Gamma^{\gamma}_{\gamma\beta} - \nabla_\gamma \delta\Gamma^{\gamma}_{\alpha\beta}. \tag{4.13}$$

We next include the variation of the Christoffel symbols:

$$\delta R_{\alpha\beta} = \frac{1}{2}(g^{\gamma\eta}\nabla_\alpha\nabla_\beta\delta g_{\gamma\eta} + g^{\gamma\eta}\nabla_\gamma\nabla_\eta\delta g_{\alpha\beta} - g^{\gamma\eta}\nabla_\gamma(\nabla_\beta\delta g_{\alpha\eta} + \nabla_\alpha\delta g_{\beta\eta})). \tag{4.14}$$

We plug this term back into our δR expression:

$$\begin{aligned}
\delta R &= \delta(g^{\alpha\beta}R_{\alpha\beta}) \qquad (4.15)\\
&= (\delta g^{\alpha\beta})R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\\
&= R_{\alpha\beta}\delta g^{\alpha\beta} + g^{\alpha\beta}(\frac{1}{2}(g^{\gamma\eta}\nabla_\alpha\nabla_\beta\delta g_{\gamma\eta} + g^{\gamma\eta}\nabla_\gamma\nabla_\eta\delta g_{\alpha\beta} - g^{\gamma\eta}\nabla_\gamma(\nabla_\beta\delta g_{\alpha\eta} + \nabla_\alpha\delta g_{\beta\eta})))\\
&= R_{\alpha\beta}\delta g^{\alpha\beta} + \nabla^\gamma\nabla_\gamma(g^{\alpha\beta}\delta g_{\alpha\beta}) - \nabla^\alpha\nabla^\beta\delta g_{\alpha\beta}.
\end{aligned}$$

If we so choose, we can make this expression look nicer by defining a 1-form ω to be

$$\omega_\alpha = g^{\gamma\eta}_{\alpha}\delta g_{\gamma\eta} - \nabla^\beta\delta g_{\alpha\beta}, \tag{4.16}$$

so we have

$$\delta R = R_{\alpha\beta}\delta g^{\alpha\beta} + \nabla^\alpha\omega_\alpha. \tag{4.17}$$

Finally, now that we have our full variation expression,

$$\begin{aligned}
\delta S &= \int_M (\delta R)\text{vol} + R\delta\text{vol} \qquad (4.18)\\
&= \int_M (R_{\alpha\beta}\delta g^{\alpha\beta} + \nabla^\alpha\omega_\alpha - \frac{1}{2}Rg_{\alpha\beta}\delta g^{\alpha\beta})\text{vol},
\end{aligned}$$

we can start deriving Einstein's equation. If we look at the part of this equation without the ω-term, we have

$$\int_M (R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta})(\delta g^{\alpha\beta})\text{vol}. \tag{4.19}$$

Remember Einstein's equation,

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 0. \tag{4.20}$$

When this equation is true, our integral is zero for all variations $\delta g^{\alpha\beta}$ which vanish outside a compact set M. Now for the remaining term, it is true that for any 1-form,

$$\nabla^\alpha\omega_\alpha = - \star d \star \omega, \tag{4.21}$$

so we are left with the following through Stokes' theorem, $\star^2 = \pm 1$,

$$-\int_M \text{vol} \wedge \star d \star \omega = \pm \int_M d \star \omega \wedge \star vol = \pm \int_M d \star \omega = 0. \tag{4.22}$$

We can now know the linearized vacuum Einstein equation. We solve using a perturbation $g + \epsilon h$ such that the new metric is a solution up to first order in ϵ. The Ricci tensor for g vanishes, since g is a solution, so the Ricci tensor of $g + \epsilon h$ is

$$\frac{\epsilon}{2}(g^{\gamma\eta}\nabla_\alpha\nabla_\beta h_{\gamma\eta} + g^{\gamma\eta}\nabla_\gamma\nabla_\eta h_{\alpha\beta} - g^{\gamma\eta}\nabla_\gamma(\nabla_\beta h_{\alpha\eta} + \nabla_\alpha h_{\beta\eta})), \tag{4.23}$$

along with higher order terms. If h is to vanish, it must satisfy the linearized Einstein equation, which we find to be

$$g^{\gamma\eta}\nabla_\alpha\nabla_\beta h_{\gamma\eta} + g^{\gamma\eta}\nabla_\gamma\nabla_\eta h_{\alpha\beta} - g^{\gamma\eta}\nabla_\gamma(\nabla_\beta h_{\alpha\eta} + \nabla_\alpha h_{\beta\eta}) = 0. \tag{4.24}$$

The linearized Einstein equation is used to study gravitational waves. Gravitational waves are interesting in that they are ripples across spacetime caused by massive objects moving at extreme accelerations.[4] This example includes neutron stars and black holes. The detection of gravitational waves offers a separate detection method to electromagnetic waves, as gravitational waves are largely unimpeded by matter. We can track ripples farther away than with light detection, which can be bent by massive celestial bodies, or even blocked by space materials.

5 ADM Formalism

Let us say we have a Lorentzian manifold M and a diffeomorphism defined

$$\phi : M \longrightarrow \mathbb{R} \times S, \tag{5.1}$$

for which $t \in \mathbb{R}$ is our time and S is space. We define a time coordinate on spacetime M as the pullback of t on $\mathbb{R} \times S$, $\tau = \phi * t$. We will say that $\Sigma \subset M$ is a slice of spacetime M if time τ is constant on Σ. We call the metric on Σ the 3-metric, or ${}^3g$. The way Σ sits in M, or the extrinsic curvature, represents the time derivative of ${}^3g$, which is K in our context, and our Cauchy data for the metric will be $({}^3g, K)$.

The Einstein tensor contains ten independent components, so, in actuality, rather than the Einstein equation, we have the Einstein equation*s*. This section is about the formulation of these ten equations, the Arnowitt-Deser-Misner (ADM) formulation. The first four equations we will consider are constraint equations for the Cauchy data. The six other equations describe the evolution of the 3-metric over time.

5.1 Extrinsic Curvature

When Σ is a spacelike slice of spacetime M, with the 3-metric, a field of timelike normal unit vectors n exists such that

$$g(n, n) = -1 \tag{5.2}$$

and $\forall\ \upsilon \in T_p\Sigma$, $g(n, \upsilon) = 0$, for which $T_p\Sigma$ is the tangent space on Σ. If we switch the sign of n, we can designate directions for the future and past. We can expand any vector $\upsilon \in T_pM$ into two components

$$\upsilon = -g(\upsilon, n)n + (\upsilon + g(\upsilon, n)n). \tag{5.3}$$

We can check that the normal component of n is just n,

$$-g(n,n)n = -(-1)n = n, \tag{5.4}$$

and that the tangential component of v is orthogonal to n,

$$g(v + g(v,n)n, n) = g(v,n) + g(v,n)g(n,n) = g(v,n) - g(v,n) = 0. \tag{5.5}$$

Then, for any vector fields u, v on Σ,

$$\begin{aligned} \nabla_u v &= -g(\nabla_u v, n)n + (\nabla_u v + g(\nabla_u v, n)n) \\ &= K(u,v)n + (\nabla_u v + g(\nabla_u v, n)n), \end{aligned} \tag{5.6}$$

where $K(u,v)$ is the extrinsic curvature. When parallel translated using the Levi-Civita connection ∇ on M, a tangent vector of Σ will fail to a certain degree to be tangent. This failure is the extrinsic curvature.

The second term is the Levi-Civita connection associated to ${}^3g$:

$${}^3\nabla_u v = \nabla_u v + g(\nabla_u v, n)n. \tag{5.7}$$

To prove this fact, we first show this expression is a connection for some $u, v, w \in \mathrm{Vect}(\Sigma)$, $f \in C^\infty(\Sigma)$, and $g(n, w) = 0$,

$$\begin{aligned} {}^3\nabla_u(fw) &= \nabla_v(fw) + g(n, \nabla_v(fw))n \\ &= v(f)w + f\nabla_v w + g(n, v(f)w)n + g(n, f\nabla_v w)n \\ &= v(f)w + f\nabla_v w + fg(n, \nabla_v w)n \\ &= v(f)w + f(\nabla_v w + g(n, \nabla_v w)n)) \\ &= v(f)w + f^3\nabla_u w. \end{aligned} \tag{5.8}$$

Next, we prove metric preservation with $g(n,w) = g(v,n) = 0$:

$$\begin{aligned} ug(v,w) &= g(\nabla_u, w) + g(v, \nabla_u w) \\ &= g(K(u,v)n +^3 \nabla_u, w) + g(v, K(u,w)n +^3 \nabla_u w) \\ &= g({}^3\nabla_u v, w) + g(v,^3 \nabla_u w). \end{aligned} \tag{5.9}$$

Lastly, we prove this result is torsion-free, using $K(u,v) = K(v,u)$:

$$\begin{aligned} {}^3\nabla_u v -^3 \nabla_v u &= \nabla_u v - K(u,v)n - \nabla_v u + K(v,u)n \\ &= \nabla_u v - \nabla_v u \\ &= [u,v]. \end{aligned} \tag{5.10}$$

A tensor depends $C^\infty(\Sigma)$-linearly on its associated vector fields, so, given that $K(u,v)$ is a symmetric tensor,

$$K(u,v) = K_{ij}u^i v^j \tag{5.11}$$

such that

$$K_{ij} = K(\partial_i, \partial_j). \tag{5.12}$$

We can show that $K(u, v)$ is $C^\infty(\Sigma)$-linear in u,

$$K(fu, v) = -g(\nabla_{fu} v, n) = -g(f\nabla_u v, n) = -fg(\nabla_u v, n) = fK(u, v), \tag{5.13}$$

and in v,

$$\begin{aligned} K(u, fv) &= -g(\nabla_u fv, n) \\ &= -g(u(f)v + f\nabla_u v, n) \\ &= -fg(\nabla_u v, n) \\ &= -fK(u, v). \end{aligned} \tag{5.14}$$

For symmetry, using that ∇ is torsion free,

$$\begin{aligned} K_{ij} - K_{ji} &= K(\partial_i, \partial_j) - K(\partial_j, \partial_i) \\ &= -g(\nabla_i \partial_j, n) + g(\nabla_j \partial_i, n) \\ &= -g(\nabla_i \partial_j - \nabla_j \partial_i, n) \\ &= -g([\partial_i, \partial_j], n) \\ &= 0. \end{aligned} \tag{5.15}$$

We can also define $K(u, v) = g(\nabla_u n, v)$, which agrees with the argument from the metric preservation portion of the proof, substituting n for v and v for w,

$$0 = ug(n, v) = g(\nabla_u v, n) + g(n, \nabla_u v). \tag{5.16}$$

We see that this definition allows for extrinsic curvature to be seen from the perspective that it is how much parallel translating in the direction of u is n forced to rotate in the direction of v.

5.2 The Gauss-Codazzi Equations

We will now show that four of Einstein's equations in 4-dimensional spacetime, the Gauss-Codazzi equations describe constraints on ${}^3g$ and K. We revisit our diffeomorphism:

$$\phi : M \longrightarrow \mathbb{R} \times S. \tag{5.17}$$

Let us define a particular vector field ∂_t on M as the pushforward by ϕ^{-1} of ∂_t on $\mathbb{R} \times S$. We will focus on the slice Σ for which $\tau = 0$. We split ∂_τ into normal and tangential components

$$\partial_\tau = -g(\partial_\tau, n)n + (\partial_\tau + g(\partial_\tau, n)n) = Nn + \vec{N}, \tag{5.18}$$

the lapse function being N and the shift vector being $\vec{N}$. We then have an expression for the unit normal,

$$n = \frac{1}{N}(\partial_\tau - \vec{N}). \tag{5.19}$$

Now, we choose a point p on Σ with local coordinates $x^0 = \tau$, x^1, x^2, x^3 and vector fields $\partial_0 = \partial_\tau$, ∂_1, ∂_2, ∂_3 tangent to Σ. The Christoffel symbols of ${}^3\nabla$ will be ${}^3\Gamma^i_{jk}$ and the Riemann tensor of ${}^3g$ will be ${}^3R^m_{ijk}$. We want R^α_{ijk} in terms of K_{ij} and 3-Riemann. We start with

$$R(\partial_i, \partial_j)\partial_k = \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k. \tag{5.20}$$

For any vector fields u, v on Σ,

$$\nabla_u v = K(u, v)n +^3 \nabla_u v \Rightarrow \nabla_i \partial_j = K_{ij} n +^3 \Gamma^m_{ij} \partial_m, \tag{5.21}$$

and, since $K(u, v) = g(\nabla_u n, v)$, $\nabla_i n = C^m_i \partial_m$.

We can now find the first term,

$$\begin{aligned}
\nabla_i \nabla_j \partial_k &= \nabla_i (K_{jk} n +^3 \Gamma^m_{jk} \partial_k) \\
&= K_{jk,i} n + K_{jk} \nabla_i n +^3 \Gamma^m_{jk,i} \partial_m +^3 \Gamma^m_{jk} \nabla_i \partial_m \\
&= K_{jk,i} n + K_{jk} K^m_i \partial_m +^3 \Gamma^m_{jk,i} \partial_m +^3 \Gamma^m_{jk} (K_{im} n +^3 \Gamma^\ell_{im} \partial_\ell) \\
&= (K_{jk,i} n +^3 \Gamma^m_{jk} K_{im}) n + K_{jk} K^m_i \partial_m + ({}^3\Gamma^m_{jk,i} +^3 \Gamma^\ell_{jk} {}^3\Gamma^m_{i\ell}) \partial_m.
\end{aligned} \tag{5.22}$$

The second term just has switched i, j indices, so completing the subtraction,

$$\begin{aligned}
R(\partial_i, \partial_j)\partial_k =& (K_{jk,i} - K_{ik,j} +^3 \Gamma^m_{jk} K_{im} -^3 \Gamma^m_{ik} K_{jm}) n \\
&+ (K_{jk} K^m_i - K_{ik} K^m_j) \partial_m \\
&+ ({}^3\Gamma^m_{jk,i} -^3 \Gamma^m_{ik,j} +^3 \Gamma^\ell_{jk} {}^3\Gamma^m_{i\ell} -^3 \Gamma^\ell_{ik} {}^3\Gamma^m_{j\ell}) \partial_m.
\end{aligned} \tag{5.23}$$

We can condense the first line to

$$({}^3\nabla_i K_{jk} -^3 \nabla_j K_{ik}) n \tag{5.24}$$

and the third line to

$${}^3R^m_{ijk} \partial_m \tag{5.25}$$

to retrieve our Gauss-Codazzi equations:

$$R(\partial_i, \partial_j)\partial_k = ({}^3\nabla_i K_j k -^3 \nabla_j K_{ik}) n + ({}^3R^m_{ijk} + K_{jk} K^m_i - K_{ik} K^m_j) \partial_m. \tag{5.26}$$

The formulae are easier to deal with if we assume the lapse is 1 and the shift is 0. If we apply dx^0 to both sides, we find the Gauss equation,

$$R^0_{ijk} =^3 \nabla_i K_{jk} -^3 \nabla_j K_{ik}, \tag{5.27}$$

or if we apply dx^m, we find the Codazzi equation,

$$R^m_{ijk} =^3 R^m_{ijk} + K_{jk} K^m_i - K_{ik} K^m_j. \tag{5.28}$$

This result says that intrinsic curvature ${}^3R^m_{ijk} = R^m_{ijk}$ when extrinsic curvature $K = 0$.

To find the four G^0_α Einstein equations, we break down the Riemann tensor symmetries:

$$R^\alpha_{\mu\alpha\nu} = -R^\alpha_{\alpha\mu\nu} = -R^\alpha_{\mu\nu\alpha} = R^\alpha_{\mu\nu\alpha} = R_{\mu\nu}. \tag{5.29}$$

If we raise the μ index, $R^\mu_\nu = R^{\mu\alpha}_{\nu\alpha}$, therefore,

$$G^{\mu}_{\nu} = R^{\mu\alpha}_{\nu\alpha} - \frac{1}{2}\delta^{\mu}_{\nu} R^{\alpha\beta}_{\alpha\beta}. \tag{5.30}$$

If $\mu = \nu = 0$, $G^0_0 = -(R^{12}_{12} + R^{23}_{23} + R^{31}_{31})$. Then, with the Codazzi equation,

$$-G^0{}_0 =^{(3)} R^{12}{}_{12} + ^{(3)}R^{23}{}_{23} + ^{(3)}R^{31}{}_{31} + (K^2{}_1K^1{}_2 - K^2{}_2K^1{}_1) + (K^3{}_2K^2{}_3 - K^3{}_3K^2{}_2) + (K^1{}_3K^3{}_1 - K^1{}_1K^3{}_3). \tag{5.31}$$

Now, the K terms are equal to $-\frac{1}{2}((K^i_i)^2 - K_{ij}K^{ij})$, so

$$^{(3)}R^{12}{}_{12} +^{(3)} R^{23}{}_{23} +^{(3)} R^{31}_{31} = \frac{1}{2}{}^3R. \tag{5.32}$$

We use these terms to find that

$$G^0{}_0 = -\frac{1}{2}({}^3R + (K^i{}_i)^2 - K_{ij}K^{ij}), \tag{5.33}$$

and, if K is a matrix,

$$G^0{}_0 = -\frac{1}{2}({}^3R + (trK)^2 - tr(K^2)) = 8\pi\kappa T^0_0, \tag{5.34}$$

which is a constraint relating the K of a Σ slice to its scalar curvature.

If we instead set $\mu = 0$ and $\nu = 1$, we ignore the second term to get

$$G^0{}_1 = R^{0\alpha}{}_{1\alpha} = R^0{}_{\alpha 1}{}^{\alpha} = R^0{}_{01}{}^0 + R^0{}_{11}{}^1 + R^0{}_{21}{}^2 + R^0{}_{31}{}^3. \tag{5.35}$$

The first two terms disappear, so $G^0_1 = R^0_{21}{}^2 + R^0_{31}{}^3$. Then, with the Gauss equation,

$$\begin{aligned} G^0_1 &= ({}^3\nabla_2 K^2_1 -^3 \nabla_1 K^2_2) + ({}^3\nabla_3 K^3_1 -^3 \nabla_1 K^3_3) \\ &= ({}^3\nabla_1 K^1_1 -^3 \nabla_1 K^j_j) - ({}^3\nabla_1 K^1_1 -^3 \nabla_j K^j_1) \\ &=^3 \nabla_j K^j_1 -^3 \nabla_1 K^j_j. \end{aligned} \tag{5.36}$$

This expression is general for all G^0_i constraints:

$$G^0_i =^3 \nabla_j K^j_i -^3 \nabla_i K^j_i = 8\pi\kappa T^0_i, \tag{5.37}$$

constraints on K for any Σ.

If we were to drop the original assumption that the lapse is 1 and the shift is 0, we would find the general constraints

$$G_{\mu\nu}n^{\mu}n^{\nu} = -\frac{1}{2}({}^3R + (trC)^2 - tr(C^2)) \tag{5.38}$$

and

$$G_{\mu i}n^{\mu} =^3 \nabla_j C^j_i -^3 \nabla_i C^j_j. \tag{5.39}$$

We have successfully found our four restraint equations.

5.3 Canonical Quantization

We can quantize gravity into particles called gravitons when gravity is weak and spacetime is approximately flat.[9] In this section, we look at how the quantum mechanics of a particle in n-dimensional space, $\mathbb{R}^n$ applies to quantum gravity. Let the particle be traveling a path $q(t)$ in this configuration space satisfying the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}, \tag{5.40}$$

for which the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 – V(q), \tag{5.41}$$

where $\dot{q}$ is the velocity, m is mass, and potential energy is V. Between the two equations, we have the equation of motion,

$$m\ddot{q} = -\nabla V(q). \tag{5.42}$$

We then have the momentum conjugate to the position,

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{5.43}$$

When a particle is in a potential, momentum $p = m\dot{q}$, and, in general, the Hamiltonian is described by

$$H(p, q) = p \cdot \dot{q} – L(q, \dot{q}).$$

For a particle in a potential,

$$H(p, q) = p^2/2m – V(q), \tag{5.44}$$

but this relationship does not hold in general relativity due to constraints.

To convert the Euler-Lagrange equations into evolutionary equations for the state over time, we first compute

$$dH = \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial q^i}dq^i \tag{5.45}$$

then, using the definition of momentum,

$$\begin{aligned} dH &= \dot{p}_i d\dot{q}^i + \dot{q}^i dp_i - \frac{\partial L}{\partial q^i}dq^i - \frac{\partial L}{\partial \dot{q}^i}d\dot{q}^i \\ &= \dot{p}_i d\dot{q}^i + \dot{q}^i dp_i - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}dq^i – p_i d\dot{q}^i \\ &= \dot{q}^i dp_i - \dot{p}_i dq^i. \end{aligned} \tag{5.46}$$

When we combine these expressions at dH, we find Hamilton's equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \dot{p}_i = \frac{\partial H}{\partial p^i}. \tag{5.47}$$

We can write a neater definition if we use the Poisson bracket as an algebra on phase space,

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} \tag{5.48}$$

and the Leibniz Law,

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \tag{5.49}$$

and get

$$\dot{q}^i = \{H, q^i\}, \tag{5.50}$$
$$\dot{p}_i = \{H, p_i\}. \tag{5.51}$$

If we consider the time derivative of observable f,

$$\begin{aligned} \frac{d}{dt} f(p, q) &= \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial p_i}\dot{p}_i \\ &= \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i} \\ &= \{H, f\}, \end{aligned} \tag{5.52}$$

we see that the Poisson bracket with the Hamiltonian gives the rate of change of the observable. The Hamiltonian generates time evolution.[1]

If we wish to quantize this particle, we need to have observables which are self-adjoint operators on $L^2(\mathbb{R}^n)$. We exchange f for $\hat{f}$ such that if $\{f, g\} = k$, $[\hat{f}, \hat{g}] = -i\hat{k}\hbar$, then let $\hbar = 1$ in our units.

If we manage to assign operators to observables in the ideal manner, we can set

$$\hat{f}_t = e^{it\hat{H}}\hat{f}e^{-it\hat{H}} \tag{5.53}$$

to find the time evolution of observables to be

$$\frac{d}{dt}\hat{f}_t = i[\hat{H}, \hat{f}_t]. \tag{5.54}$$

In the case of a free particle in potential $V = 0$, position and momentum have the Poisson brackets $p_j, q^k = \delta^k_j$ and $p_j, p_k = q^j, q^k = 0$. Now, our operators will be

$$(\hat{q}^j\psi)(x) = x^j\psi(x), \tag{5.55}$$
$$(\hat{p}_j\psi)(x) = -i\partial_j\psi(x), \tag{5.56}$$

and our canonical commutation relations

$$[\hat{p}_j, \hat{q}^k] = -i\delta^k_j, \tag{5.57}$$
$$[\hat{p}_j, \hat{p}_k] = [\hat{q}^j, \hat{q}^k] = 0, \tag{5.58}$$

while converting our original Hamiltonian to $\hat{H} = \frac{\hat{p}^2}{2m}$.

To apply this concept to general relativity, we consider spacetime M to be diffeomorphic to $\mathbb{R} \times S$ with our fixed spacelike slice Σ. We will only work in the vacuum Einstein equation so as not to make things more complicated. Instead of q, we use ${}^3g_{ij} ={}^3 q_{ij}$, and, instead of $\mathbb{R}^n$, we use $\mathrm{Met}(\Sigma)$ as the configuration space for gravity. If we go back to our K_{ij} derivations, in terms of operators, we find that

$$K_{ij} = \frac{1}{2}N^{-1}(\hat{q}_{ij} -^3 \nabla_i N_j -^3 \nabla_j N_i), \tag{5.59}$$

and

$$L = R\sqrt{-\det g}d^4x, \tag{5.60}$$

which we will alter to have

$$L = R\sqrt{-\det g} = q^{1/2}NR = q^{1/2}N(^3R + tr(K^2)–(trK)^2). \tag{5.61}$$

From the above forms of the Lagrangian and extrinsic curvature,

$$p^{ij} = \frac{\partial L}{\partial \dot{q}_{ij}} = q^{1/2}(K^{ij}–tr(K)q^{ij}). \tag{5.62}$$

When we integrate the Hamiltonian density over the slice, we find the Hamiltonian:

$$H = \int_\Sigma H(p^{ij}, q_{ij})d^3x = \int_\Sigma p_{ij}\dot{q}^{ij} - Ld^3x \tag{5.63}$$

when in a compact space. Since we are in the compact case, we can ignore total divergences and get the form

$$H = q^{1/2}(NC + N^iC_i), \tag{5.64}$$

for which $C = -^3R + q^{-1}(tr(p^2) - \frac{1}{2}tr(p)^2) = -2G_{\mu\nu}n^\mu n^\nu$ and $C_i = -2^3\nabla^j(q^{-1/2}p_{ij}) = -2G_{\mu i}n^\mu$.

Because of these C constraints, the vacuum equation implies that the Hamiltonian density disappears. This result is because $C = C_i = 0$ are the four constraint Einstein equations.

The phase space may be the space of the cotangent bundle $T^*\text{Met}(\Sigma)$, but not all position-momentum coefficient pairs represent allowed states.[1] The constraints provide us with a phase space of allowed states called the physical phase space,

$$X = \{C = C_i = 0\} \subset T^*\text{Met}(\Sigma), \tag{5.65}$$

over which H vanishes. Our Hamiltonian equations, however, still provide nontrivial dynamics. We can perform variation calculations on the Poisson brackets of p_{ij} and q^{ij}:

$$\{p^{ij}(x), q_{k\ell}(y)\} = (\delta^i_k\delta^j_\ell + \delta^i_\ell\delta^j_k)\delta^{(3)}(x-y) \tag{5.66}$$

$$\{p^{ij}(x), p^{k\ell}(y)\} = 0 \tag{5.67}$$

$$\{q_{ij}(x), q_{k\ell}(y)\} = 0. \tag{5.68}$$

We then use Einstein's equations $G_{ij} = 0$ of the second time derivative of $^3q_{ij}$ to get our six time evolutionary equations when i, j = 1,2,3:

$$\dot{q}^{ij} = \{H, q^{ij}\}, \tag{5.69}$$

$$\dot{p}_{ij} = \{H, p_{ij}\}, \tag{5.70}$$

the expansions of which are the non-trivial

$$\dot{q}_{ij} = 2q^{-1/2}N(p_{ij} - \frac{1}{2}p^k_k q_{ij}) + 2^3\nabla_{[i}N_{j]}, \tag{5.71}$$

$$\dot{p}^{ij} = -Nq^{-1/2}(^3R^{ij} - \frac{1}{2}^3 R^{ij}q^{ij}) + \frac{1}{2}Nq^{-1/2}q^{ij}(p_{ab}p^{ab} - \frac{1}{2}(p^a_a)^2) \quad (5.72)$$
$$-2Nq^{-1/2}(p^{ia}p^j_a - \frac{1}{2}p^a_a p^{ij}) + q^{1/2}(\nabla^i\nabla^j N - q^{ij3}\nabla^{a3}\nabla_a N)$$
$$+ q^{1/2}\nabla_a(q^{-1/2}N^a p^{ij}) - 2p^{a[i3}\nabla_a N^{j]}$$

When we attempt to quantize gravity, the first problem we face is defining a Hilbert space. Since Met(Σ) is infinite-dimensional, we cannot find an obvious square-integrable function to use, but we pretend that L^2(Met(Σ) works regardless. Our substitute operator for the 3^g, for $g \in$ Met(Σ), and x a point on Σ,

$$(\hat{q_{ij}}(x)\psi)(q) = g_{ij}(x)\psi(q), \quad (5.73)$$

with our momentum operator as

$$(\hat{p}_{ij}(x)\psi)(q) = -i\frac{\partial}{\partial q_{ij}(x)}\psi(q). \quad (5.74)$$

These forms give us our canonical commutation relations,

$$[\hat{p}^{ij}(x), \hat{q}_{k\ell}(y)] = -i(\delta^i_k\delta^j_\ell + \delta^i_\ell\delta^j_k)\delta^{(3)}(x,y) \quad (5.75)$$
$$[\hat{p}^{ij}(x), \hat{p}^{k\ell}(y)] = 0 \quad (5.76)$$
$$[\hat{q}_{ij}(x), \hat{q}_{k\ell}(y)] = 0. \quad (5.77)$$

When trying to quantize the Hamiltonian, we run into an operator ordering problem, since, if we replace q_{ij}, p^{ij} with $\hat{q_{ij}}, \hat{p^{ij}}$, we are confronted with the fact that these expressions do not commute. The constraints, therefore, are difficult to achieve. Ideally, we would find constraints satisfying

$$[\hat{C}(\vec{N}), \hat{C}(\vec{N}')] = -i\hat{C}([\vec{N}, \vec{N}']) \quad (5.78)$$
$$[\hat{C}(\vec{N}, \hat{C}(N')] = -i\hat{C}(\vec{N}N') \quad (5.79)$$
$$[\hat{C}(N), \hat{C}(N')] = -i\hat{C}((N\partial^i N' - N'\partial^i N)\partial_i), \quad (5.80)$$

so we will act forward as though we could. We would then find that our quantum theory Hamiltonian is

$$\hat{H} = \int_\Sigma (N\hat{C} + N^i\hat{C}_i)q^{1/2}d^3x. \quad (5.81)$$

If we use Dirac algebra, for which $\psi \in L^2$Met(Σ) is a physical state, it must satisfy the quantum restraints

$$\hat{C}(N)\psi = \hat{C}(\vec{N})\psi = 0. \quad (5.82)$$

We could also say that, for all $N, \vec{N}$, the Wheeler-DeWitt equation holds with $\hat{H}\psi = 0$. In either case, we can force the Hamiltonian to disappear in the physical phase space. Unfortunately, few solutions have been found for this Wheeler-DeWitt equation. In addition to this fact, if we were to find physical states, the physical phase space would be

$$H_{phys} = \{\psi : \forall N, \vec{N}\hat{C}(N)\psi = \hat{C}(\vec{N})\psi = 0\}, \tag{5.83}$$

of which we cannot expect the physically relevant inner product to agree with the $L^2\text{Met}(\Sigma)$ inner product. Since our Hamiltonian vanishes on H_{phys}, any operator will commute with it, leading to the revelation that these operators do not change with time:

$$\frac{d}{dt}A_t = i[\hat{H}, A_t] = 0. \tag{5.84}$$

This result is because H_{phys} describes quantum gravity for all time, rather than with time dependency. Therefore, quantum gravity faces many roadblocks to drawing a connection between gravity and the three other fundamental forces. This topic this, however, a growing field. For example, there has been achieved quantum entanglement of diamonds and - possibly - waterbears. It is believed that this approach is the future to finding a method of experiencing the quantization of gravity.

References

[1] John Baez, Javier P. Muniain (1994) *Gauge Fields, Knots and Gravity*, World Scientific.

[2] Alessandro Danieli (2019) *ADM formalism: a Hamiltonian approach to General Relativity*, Universitas Studiorum Mediolanensis.

[3] Andrew Hamilton (2019) *Schwarzschild Geometry*, https://jila.colorado.edu/ ajsh/bh/schwp.html.

[4] LIGO Caltech website, https://www.ligo.caltech.edu/page/gw-sources.

[5] Richard S. Millman, George D. Parker (1977) *Elements of Differential Geometry*, Prentice-Hall.

[6] Aula Raul (2018) *Solutions to the Friedmann Equation*, http://fma.if.usp.br.

[7] Joel W. Robbin, Dietmar A. Salamon (2021) *Introduction to Differential Geometry*, Lecture notes.

[8] Iva Stavrov (2020) *Curvature of Space and time, with an Introduction to Geometric Analysis*, American Mathematical Society Institute for Advanced Study.

[9] Hajime Urakawa (1993) *Calculus of Variations and Harmonic Maps*, American Mathematical Society.

[10] Robert M. Wald (1984) *General Relativity*, The University of Chicago Press.